MAA OMWATI DEGREE COLLEGE

HASSANPUR (PALWAL)

SUBJECT – ELECTRICITY & MAGNETISM

CLASS- B.SC II SEM

SESSION- 2024-25

Vector Operators: Grad, Div and Curl

In the first lecture of the second part of this course we move more to consider properties of fields. We introduce three field operators which reveal interesting collective field properties, viz.

- the **gradient** of a scalar field,
- the **divergence** of a vector field, and
- the **curl** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

In Lecture 6 we will look at combining these vector operators.

5.1 The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how.

If $U(\mathbf{r}) = U(x, y, z)$ is a scalar field, ie a scalar function of position $\mathbf{r} = [x, y, z]$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\text{grad}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} . \tag{5.1}$$

It is usual to ne the **vector operator** which is called "del" c ıabla"

$$\nabla = \hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} \;+\; \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} \;+\; \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\ . \tag{5.2}$$

Then

$$\mathrm{grad}U \equiv \nabla U\ . \tag{5.3}$$

Note immediately that ∇U is a vector field!

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

♣ Worked examples of gradient evaluation

1. $U = x^2$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\mathbf{\imath}} \;+\; \frac{\partial}{\partial y}\hat{\mathbf{\jmath}} \;+\; \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) x^2 = 2x\hat{\mathbf{\imath}}\ . \tag{5.4}$$

2. $U = r^2$

$$r^2 \;=\; x^2 + y^2 + z^2 \tag{5.5}$$

$$\Rightarrow \nabla U \;=\; \left(\frac{\partial}{\partial x}\hat{\mathbf{\imath}} \;+\; \frac{\partial}{\partial y}\hat{\mathbf{\jmath}} \;+\; \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)(x^2 + y^2 + z^2) \tag{5.6}$$

$$=\; 2x\hat{\mathbf{\imath}} + 2y\hat{\mathbf{\jmath}} + 2z\hat{\mathbf{k}} \;=\; 2\,\mathbf{r}\ . \tag{5.7}$$

3. $U = \mathbf{c}\cdot\mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \nabla U = \left(\hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} \;+\; \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} \;+\; \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\mathbf{\imath}} + c_2\hat{\mathbf{\jmath}} + c_3\hat{\mathbf{k}} = \mathbf{c}\ . \tag{5.8}$$

4. $U = U(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$. **NB NOT** $U(\mathbf{r})$.

   U is a function of r alone so dU/dr exists. As $U = U(x, y, z)$ also,

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial U}{\partial y} = \frac{dU}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial U}{\partial z} = \frac{dU}{dr}\frac{\partial r}{\partial z}\ . \tag{5.9}$$

$$\Rightarrow \nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial U}{\partial y}\hat{\mathbf{\jmath}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \quad = \frac{dU}{dr}\left(\frac{\partial r}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial r}{\partial y}\hat{\mathbf{\jmath}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right) \tag{5.10}$$

But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z.

$$\Rightarrow \nabla U = \frac{dU}{dr}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right) = \frac{dU}{dr}\left(\frac{\mathbf{r}}{r}\right) \quad . \tag{5.11}$$

5.2 The significance of grad

If our current position is $\mathbf{r}$ in some scalar field U (Fig. 5.1(a)), and we move an infinitesimal distance $d\mathbf{r}$, we know that the change in U is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ . \tag{5.12}$$

But we know that $d\mathbf{r} = (\hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz)$ and $\nabla U = (\hat{\mathbf{i}}\partial U/\partial x + \hat{\mathbf{j}}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in U is also given by the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ . \tag{5.13}$$

Now divide both sides by ds

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ . \tag{5.14}$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.

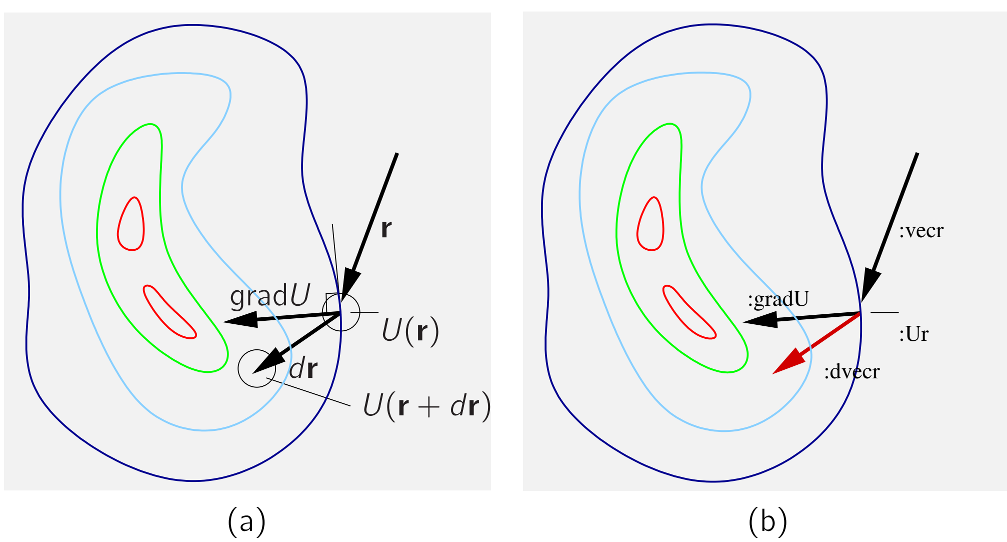


Figure 5.1: The directional derivative: The rate of change of U wrt distance in direction $\hat{\mathbf{d}}$ is $\nabla U \cdot \hat{\mathbf{d}}$.

This result can be paraphrased (Fig. 5.1(b)) as:

- grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that

- At any point P, grad$U$ points in the direction of greatest change of U at P, and has magnitude equal to the rate of change of U wrt distance in that direction.

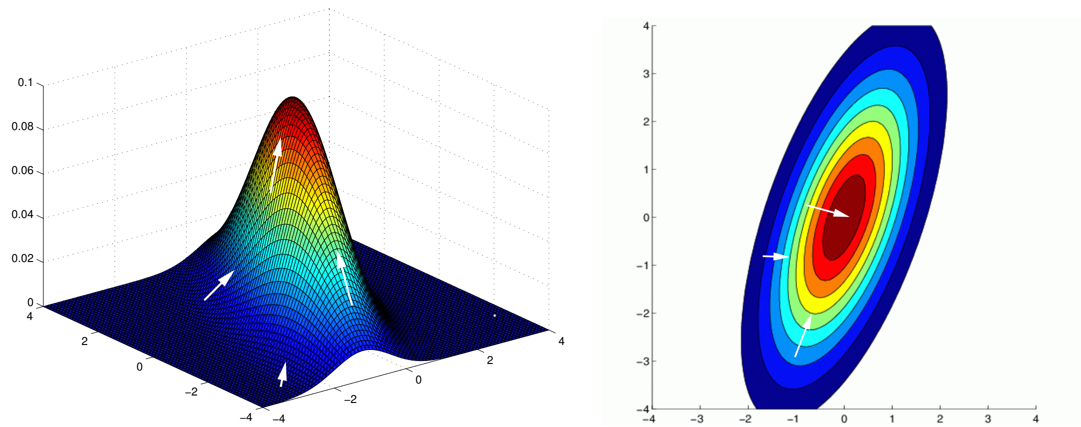


Figure 5.2: ∇U is in the direction of greatest (positive!) change of U wrt distance. (Positive $\Rightarrow$"uphill".)

Another nice property emerges if we think of a surface of constant U – that is the locus (x, y, z) for $U(x, y, z) = \text{constant}$. If we move a tiny amount within that iso-U surface, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 \,. \tag{5.15}$$

But $d\mathbf{r}/ds$ is a tangent to the surface, so this result shows that

- grad$U$ is everywhere NORMAL to a surface of constant U.

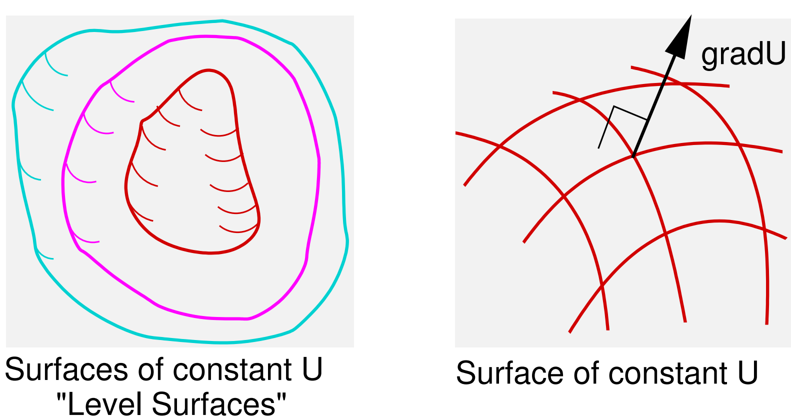


Figure 5.3: grad$U$ is everywhere NORMAL to a surface of constant U.

5.3 The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

If $\mathbf{a}(x, y, z)$ is a vector function of position in 3 dimensions, that is $\mathbf{a} = a_1\hat{\mathbf{\imath}} + a_2\hat{\mathbf{\jmath}} + a_3\hat{\mathbf{k}}$, then its divergence at any point is defined in Cartesian co-ordinates by

$$\operatorname{div}\mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \tag{5.16}$$

We can write this in a simplified notation using a scalar product with the ∇ vector differential operator:

$$\operatorname{div}\mathbf{a} = \left(\hat{\mathbf{\imath}}\frac{\partial}{\partial x} + \hat{\mathbf{\jmath}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\mathbf{a} = \nabla\cdot\mathbf{a} \tag{5.17}$$

Notice that the divergence of a vector field is a scalar field.

♣ Examples of divergence evaluation

	$\mathbf{a}$	$\operatorname{div}\mathbf{a}$
1)	$x\hat{\mathbf{\imath}}$	1
2)	$\mathbf{r}(= x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}})$	3
3)	$\mathbf{r}/r^3$	0
4)	$r\mathbf{c}$, for $\mathbf{c}$ constant	$(\mathbf{r}\cdot\mathbf{c})/r$

We work through example 3).

The x component of $\mathbf{r}/r^3$ is $x.(x^2 + y^2 + z^2)^{-3/2}$, and we need to find $\partial/\partial x$ of it.

$$\begin{aligned}\frac{\partial}{\partial x}x.(x^2+y^2+z^2)^{-3/2} &= 1.(x^2+y^2+z^2)^{-3/2} + x\frac{-3}{2}(x^2+y^2+z^2)^{-5/2}.2x \\ &= r^{-3}\left(1 - 3x^2r^{-2}\right) .\end{aligned} \tag{5.18}$$

The terms in y and z are similar, so that

$$\begin{aligned}\operatorname{div}(\mathbf{r}/r^3) &= r^{-3}\left(3 - 3(x^2+y^2+z^2)r^{-2}\right) = r^{-3}(3-3) \\ &= 0\end{aligned} \tag{5.19}$$

5.4 The significance of div

Consider a typical vector field, water flow, and denote it by $\mathbf{a}(\mathbf{r})$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{a}$ per unit time.

Now take an infinitesimal volume element dV and figure out the balance of the flow of $\mathbf{a}$ in and out of dV.

To be specific, consider the volume element $dV = dxdydz$ in Cartesian co-ordinates, and think first about the face of area $dxdz$ perpendicular to the y axis and facing outwards in the negative y direction. (That is, the one with surface area $d\mathbf{S} = -dxdz\hat{\mathbf{j}}$.)

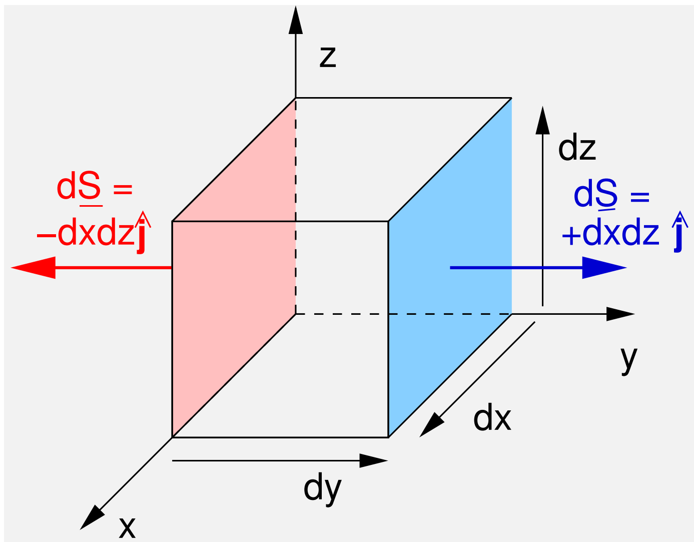


Figure 5.4: Elemental volume for calculating divergence.

The component of the vector $\mathbf{a}$ normal to this face is $\mathbf{a}\cdot\hat{\mathbf{j}} = a_y$, and is pointing inwards, and so the its contribution to the OUTWARD flux from this surface is

$$\mathbf{a}\cdot d\mathbf{S} \quad = \quad -a_y(x, y, z)dzdx\ , \tag{5.20}$$

(By the way, flux here denotes mass per unit time.)

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along y by an amount dy, so that this OUTWARD amount is

$$a_y(x, y+dy, z)dzdx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right) dxdz \tag{5.21}$$

The total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y}dydxdz = \frac{\partial a_y}{\partial y}dV \tag{5.22}$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) dV = \nabla\cdot\mathbf{a}\ \ dV \tag{5.23}$$

So we see that

> The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

(NB: The above does not constitute a rigorous proof of the assertion because we have not proved that the quantity calculated is independent of the co-ordinate system used, but it will suffice for our purposes.)

5.5 The Laplacian: div (grad$U$) of a scalar field

Recall that grad$U$ of *any* scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute div (grad$U$), even if we don't know what it means yet.

Here is where the ∇ operator starts to be really handy.

$$\begin{aligned}
\nabla\cdot(\nabla U) &= \left(\hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} + \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} + \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)U\right) && (5.24)\\
&= \left(\left(\hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} + \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{\imath}}\,\frac{\partial}{\partial x} + \hat{\mathbf{\jmath}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)\right)U && (5.25)\\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U && (5.26)\\
&= \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) && (5.27)\\
& && (5.28)
\end{aligned}$$

This last expression occurs frequently in engineering science (you will meet it next in solving Laplace's Equation in partial differential equations). For this reason, the operator ∇^2 is called the "Laplacian"

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U \qquad (5.29)$$

Laplace's equation itself is

$$\nabla^2 U = 0 \qquad (5.30)$$

♣ Examples of $\nabla^2 U$ evaluation

	U	$\nabla^2 U$
1)	$r^2 (= x^2 + y^2 + z^2)$	6
2)	xy^2z^3	$2xz^3 + 6xy^2z$
3)	$1/r$	0

Let's prove example (3) (which is particularly significant – can you guess why?).

$$1/r = (x^2 + y^2 + z^2)^{-1/2} \tag{5.31}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x} - x.(x^2 + y^2 + z^2)^{-3/2} \tag{5.32}$$

$$= -(x^2 + y^2 + z^2)^{-3/2} + 3x.x.(x^2 + y^2 + z^2)^{-5/2} \tag{5.33}$$

$$= (1/r^3)(-1 + 3x^2/r^2) \tag{5.34}$$

Adding up similar terms for y and z

$$\nabla^2 \frac{1}{r} = \frac{1}{r^3}\left(-3 + 3\frac{(x^2 + y^2 + x^2)}{r^2}\right) = 0 \tag{5.35}$$

5.6 The curl of a vector field

So far we have seen the operator ∇ applied to a scalar field ∇U; and dotted with a vector field $\nabla \cdot \mathbf{a}$.

We are now overwhelmed by an irrestible temptation to

- cross it with a vector field $\nabla \times \mathbf{a}$

This gives the **curl of a vector field**

$$\nabla \times \mathbf{a} \equiv \text{curl}(\mathbf{a}) \tag{5.36}$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad \text{(remember it this way)} \tag{5.37}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \tag{5.38}$$

♣ Examples of curl evaluation

	$\mathbf{a}$	$\nabla\times\mathbf{a}$
1)	$-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}$	$2\hat{\mathbf{k}}$
2)	$x^2y^2\hat{\mathbf{k}}$	$2x^2y\hat{\mathbf{i}}-2xy^2\hat{\mathbf{j}}$

5.7 The signficance of curl

Perhaps the first example gives a clue. The field $\mathbf{a}=-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}$ is sketched in Figure 5.5. (It is the field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega}=[0,0,1]$.) This field has a curl of $2\hat{\mathbf{k}}$, which is in the r-h screw sense out of the page. You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a}\cdot d\mathbf{r}$.

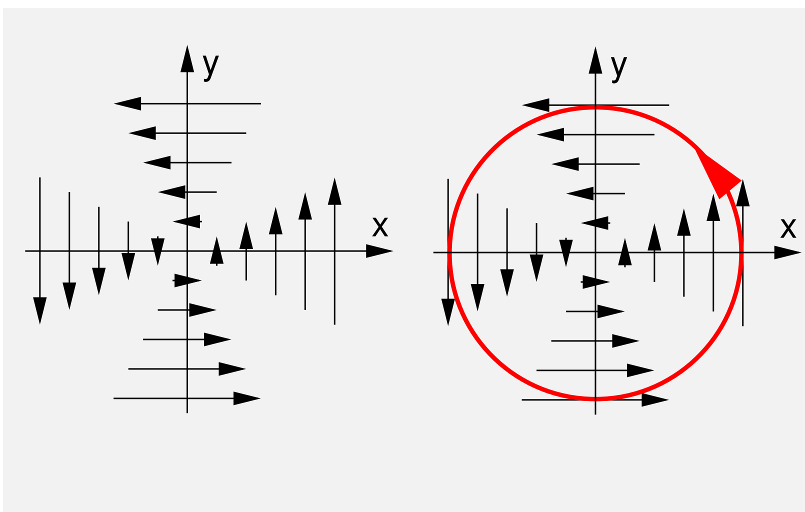


Figure 5.5: A rough sketch of the vector field $-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}$.

In fact curl is closely related to the line integral around a loop.

> The **circulation** of a vector $\mathbf{a}$ round any closed curve C is defined to be $\oint_C \mathbf{a}\cdot d\mathbf{r}$ and the **curl** of the vector field $\mathbf{a}$ represents the **vorticity**, or **circulation per unit area**, of the field.

Our proof uses the small rectangular element dx by dy shown in Figure 5.6.

Consider the circulation round the perimeter of a rectangular element.

The fields in the x direction at the bottom and top are

$$a_x(x,y,z) \qquad \text{and} \qquad a_x(x,y+dy,z)=a_x(x,y,z)+\frac{\partial a_x}{\partial y}dy, \tag{5.39}$$

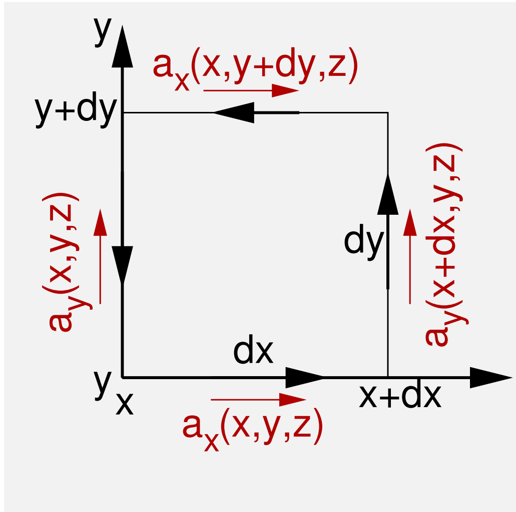


Figure 5.6: A small element used to calculate curl.

and the fields in the y direction at the left and right are

$$a_y(x,y,z) \qquad \text{and} \qquad a_y(x+dx,y,z) = a_y(x,y,z) + \frac{\partial a_y}{\partial x}dx \tag{5.40}$$

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows, where the minus signs take account of the path being opposed to the field:

$$\begin{aligned} dC &= +[a_x\ dx] + [a_y(x+dx,y,z)\ dy] - [a_x(x,y+dy,z)\ dx] - [a_y\ dy] \quad (5.41) \\ &= +[a_x\ dx] + \left[\left(a_y + \frac{\partial a_y}{\partial x}dx\right)dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y}dy\right)dx\right] - [a_y\ dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy \\ &= (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

where $d\mathbf{S} = dxdy\hat{\mathbf{k}}$.

NB: Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.

5.8 Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be **solenoidal**.
- A vector field with zero curl is said to be **irrotational**.
- A scalar field with zero gradient is said to be, er, **constant**.

Lecture 6

Vector Operator Identities

In this lecture we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

There could be a cottage industry inventing vector identities. HLT contains a lot of them. So why not leave it at that?

First, since grad, div and curl describe key aspects of vectors fields, they arise often in practice, and so the identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.

Secondly, they help to identify other practically important vector operators. So, although this material is a bit dry, the relevance of the identities should become clear later in other Engineering courses.

6.1 Identity 1: curl grad $U = 0$

$$\begin{aligned}\nabla \times \nabla U &= \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\ &= \hat{\mathbf{\imath}}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) + \hat{\mathbf{\jmath}}\,() + \hat{\mathbf{k}}\,() \\ &= \mathbf{0} \;, \end{aligned} \tag{6.1}$$

as $\partial^2/\partial y \partial z = \partial^2/\partial z \partial y$.

Note that the output is a null *vector*.

6.2 Identity 2: div curl $\mathbf{a} = 0$

$$\begin{aligned}\nabla\cdot\nabla\times\mathbf{a} &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\ &= \frac{\partial^2 a_z}{\partial x\partial y} - \frac{\partial^2 a_y}{\partial x\partial z} - \frac{\partial^2 a_z}{\partial y\partial x} + \frac{\partial^2 a_x}{\partial y\partial z} + \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_x}{\partial z\partial y} \\ &= 0\end{aligned} \tag{6.2}$$

6.3 Identity 3: div and curl of $U\mathbf{a}$

Suppose that $U(\mathbf{r})$ is a scalar field and that $\mathbf{a}(\mathbf{r})$ is a vector field and we are interested in the product $U\mathbf{a}$. This is a vector field, so we can compute its divergence and curl. For example the density $\rho(\mathbf{r})$ of a fluid is a scalar field, and the instantaneous velocity of the fluid $\mathbf{v}(\mathbf{r})$ is a vector field, and we are probably interested in mass flow rates for which we will be interested in $\rho(\mathbf{r})\mathbf{v}(\mathbf{r})$.

The divergence (a scalar) of the product $U\mathbf{a}$ is given by:

$$\begin{aligned}\nabla\cdot(U\mathbf{a}) &= U(\nabla\cdot\mathbf{a}) + (\nabla U)\cdot\mathbf{a} \\ &= U\mathrm{div}\mathbf{a} + (\mathrm{grad}U)\cdot\mathbf{a}\end{aligned} \tag{6.3}$$

In a similar way, we can take the curl of the vector field $U\mathbf{a}$, and the result should be a vector field:

$$\nabla\times(U\mathbf{a}) = U\nabla\times\mathbf{a} + (\nabla U)\times\mathbf{a} \quad . \tag{6.4}$$

6.4 Identity 4: div of $\mathbf{a}\times\mathbf{b}$

Life quickly gets trickier when vector or scalar products are involved: For example, it is not *that* obvious that

$$\mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathrm{curl}\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathrm{curl}\mathbf{b} \tag{6.5}$$

To show this, use the determinant:

$$\begin{aligned}\begin{vmatrix} \partial/\partial x_i & \partial/\partial x_j & \partial/\partial x_k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= \frac{\partial}{\partial x}[a_y b_z - a_z b_y] + \frac{\partial}{\partial y}[a_z b_x - a_x b_z] + \frac{\partial}{\partial z}[a_x b_y - a_y b_x] \\ &= \ldots \text{ bash out the products } \ldots \\ &= \mathrm{curl}\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot(\mathrm{curl}\mathbf{b})\end{aligned} \tag{6.6}$$

6.5 Identity 5: curl($\mathbf{a} \times \mathbf{b}$)

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix} \tag{6.7}$$

so the $\hat{\mathbf{i}}$ component is

$$\frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z) \tag{6.8}$$

which can be written as the sum of four terms:

$$a_x \left(\frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} \right) - b_x \left(\frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) + \left(b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z} \right) a_x - \left(a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right) b_x \tag{6.9}$$

Adding $a_x(\partial b_x/\partial x)$ to the first of these, and subtracting it from the last, and doing the same with $b_x(\partial a_x/\partial x)$ to the other two terms, we find that (you should of course check this):

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + [\mathbf{b} \cdot \nabla]\mathbf{a} - [\mathbf{a} \cdot \nabla]\mathbf{b} \tag{6.10}$$

where $[\mathbf{a} \cdot \nabla]$ can be regarded as new, and very useful, scalar differential operator.

6.6 Definition of the operator $[\mathbf{a} \cdot \nabla]$

This is a *scalar operator*, but it can obviously can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field:

$$[\mathbf{a} \cdot \nabla] \equiv \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] \quad . \tag{6.11}$$

6.7 Identity 6: curl(curl$\mathbf{a}$) for you to derive

The following important identity is stated, and left as an exercise:

$$\text{curl}(\text{curl}\mathbf{a}) = \text{graddiv}\mathbf{a} - \nabla^2 \mathbf{a} \tag{6.12}$$

where

$$\nabla^2 \mathbf{a} = \nabla^2 a_x \hat{\mathbf{i}} + \nabla^2 a_y \hat{\mathbf{j}} + \nabla^2 a_z \hat{\mathbf{k}} \tag{6.13}$$

♣ Example of Identity 6: electromagnetic waves

Q: James Clerk Maxwell established a set of four vector equations which are fundamental to working out how eletromagnetic waves propagate. The entire telecommunications industry is built on these.

$$\begin{aligned} \text{div}\mathbf{D} &= \rho \\ \text{div}\mathbf{B} &= 0 \\ \text{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\ \text{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \end{aligned} \tag{6.14}$$

In addition, we can assume the following, which should all be familiar to you:
$\mathbf{B} = \mu_r\mu_0\mathbf{H}$, $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$,
where all the scalars are constants.

Now show that in a material with zero free charge density, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field $\mathbf{E}$ must be a solution of the wave equation

$$\nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0(\partial^2\mathbf{E}/\partial t^2) \; . \tag{6.15}$$

A: First, a bit of respect. Imagine you are the first to do this — this is a tingle moment.

$$\begin{aligned} \text{div}\mathbf{D} &= \text{div}(\epsilon_r\epsilon_0\mathbf{E}) = \epsilon_r\epsilon_0\text{div}\mathbf{E} = \rho = 0 \quad \Rightarrow \text{div}\mathbf{E} = 0. \quad (a) \\ \text{div}\mathbf{B} &= \text{div}(\mu_r\mu_0\mathbf{H}) = \mu_r\mu_0\text{div}\mathbf{H} = 0 \quad \Rightarrow \text{div}\mathbf{H} = 0 \quad (b) \\ \text{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \quad (c) \\ \text{curl}\mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = 0 + \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t) \quad (d) \end{aligned} \tag{6.16}$$

Earlier (Identity 6) you found that curlcurl = graddiv $-\nabla^2$ and hence, using (c),

$$\text{curlcurl}\mathbf{E} = \text{graddiv}\mathbf{E} - \nabla^2\mathbf{E} = \text{curl}\left(-\mu_r\mu_0(\partial\mathbf{H}/\partial t)\right) \tag{6.17}$$

so interchanging the order of partial differentation, and using (a) div$\mathbf{E} = 0$:

$$\begin{aligned} -\nabla^2\mathbf{E} &= -\mu_r\mu_0\frac{\partial}{\partial t}(\text{curl}\mathbf{H}) \\ &= -\mu_r\mu_0\frac{\partial}{\partial t}\left(\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \\ \Rightarrow \nabla^2\mathbf{E} &= \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \end{aligned} \tag{6.18}$$

This equation is actually three equations, one for each component:

$$\nabla^2 E_x = \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{6.19}$$

and so on for E_y and E_z.

6.8 Grad, div, curl and ∇^2 in curvilinear co-ordinate systems

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system by making use of the h factors which were introduced in Lecture 4.

REMINDERS: The unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u} \frac{\partial \mathbf{r}}{\partial u} \tag{6.20}$$

where $\mathbf{r}$ is the position vector, and

$$h_u = \left| \frac{\partial \mathbf{r}}{\partial u} \right| \tag{6.21}$$

is the metric coefficient. Similar expressions apply for the other co-ordinate directions. Then

$$d\mathbf{r} = h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw \ . \tag{6.22}$$

6.9 Grad in curvilinear coordinates

Noting that $U = U(\mathbf{r})$ and $U = U(u, v, w)$, and using the properties of the gradient of a scalar field obtained previously

$$\nabla U \cdot d\mathbf{r} = dU = \frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial w} dw \tag{6.23}$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial w} dw \tag{6.24}$$

The only way this can be satisfied for independent du, dv, dw is when

$$\boxed{\nabla U = \frac{1}{h_u} \frac{\partial U}{\partial u} \hat{\mathbf{u}} + \frac{1}{h_v} \frac{\partial U}{\partial v} \hat{\mathbf{v}} + \frac{1}{h_w} \frac{\partial U}{\partial w} \hat{\mathbf{w}}} \tag{6.25}$$

6.10 Divergence in curvilinear coordinates

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear co-ordinates by making use of the flux property.

We consider an element of volume dV. If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order in small quantities) and

$$dV = h_u\ h_v\ h_w\ \ du\ dv\ dw\ \ . \tag{6.26}$$

However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of u, v and w in general.

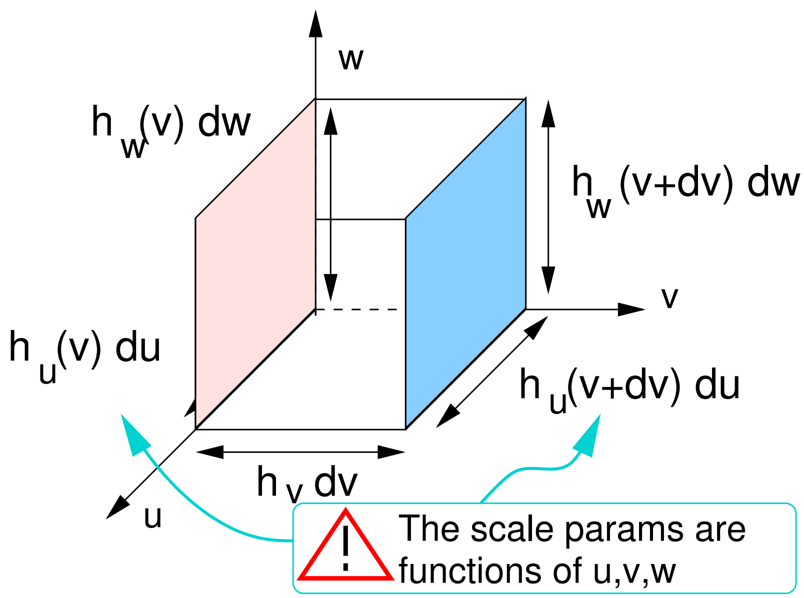


Figure 6.1: Elemental volume for calculating divergence in orthogonal curvilinear coordinates

So the net efflux from the two faces in the $\hat{\mathbf{v}}$ direction shown in Figure 6.1 is

$$\begin{aligned} &= \left[a_v + \frac{\partial a_v}{\partial v} dv\right]\left[h_u + \frac{\partial h_u}{\partial v} dv\right]\left[h_w + \frac{\partial h_w}{\partial v} dv\right] du dw - a_v h_u h_w du dw \\ &= \frac{\partial (a_v h_u h_w)}{\partial v} du dv dw \end{aligned} \tag{6.27}$$

which is easily shown by multiplying the first line out and dropping second order terms (i.e. $(dv)^2$).

By definition div is the net efflux per unit volume, so summing up the other faces:

$$\begin{aligned}\text{div}\mathbf{a}\ dV &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)\ dudvdw \\ \Rightarrow \text{div}\mathbf{a}\ h_u h_v h_w\ dudvdw &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)\ dudvdw\end{aligned}$$

So, finally,

$$\text{div}\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right) \tag{6.28}$$

6.11 Curl in curvilinear coordinates

Recall from Lecture 5 that we computed the z component of curl as the circulation per unit area from

$$dC = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy \tag{6.29}$$

By analogy with our derivation of divergence, you will realize that for an orthogonal curvilinear coordinate system we can write the area as $h_u h_v dudw$. But the opposite sides are no longer quite of the same length. The lower of the pair in Figure 6.2 is length $h_u(v)du$, but the upper is of length $h_u(v + dv)du$

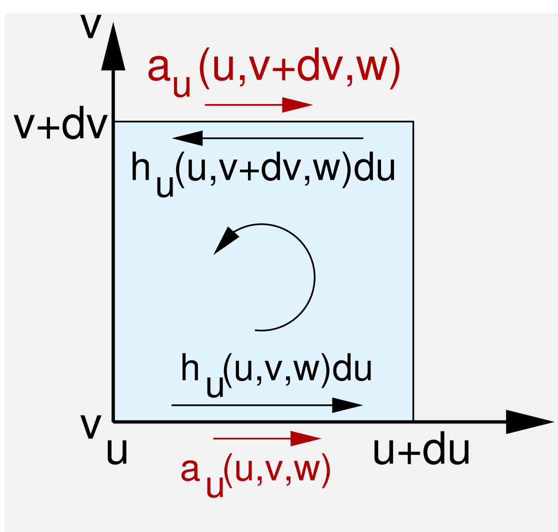


Figure 6.2: Elemental loop for calculating curl in orthogonal curvilinear coordinates

Summing this pair gives a contribution to the circulation

$$a_u(v)h_u(v)du - a_u(v+dv)h_u(v+dv)du = -\frac{\partial(h_u a_u)}{\partial v}dvdu \tag{6.30}$$

and together with the other pair:

$$dC = \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right)dudv \tag{6.31}$$

So the circulation per unit area is

$$\frac{dC}{h_u h_v dudv} = \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right) \tag{6.32}$$

and hence curl is

$$\begin{aligned}\text{curl}\mathbf{a}(u,v,w) \;=\; & \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} \;+ \\ & \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} \;+ \\ & \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}}\end{aligned} \tag{6.33}$$

You should check that this can be written as

Curl in curvilinear coords:

$$\text{curl}\mathbf{a}(u,v,w) = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix} \tag{6.34}$$

6.12 The Laplacian in curvilinear coordinates

Substitution of the components of grad$U$ into the expression for div**a** immediately (!*?) gives the following expression for the Laplacian in general orthogonal co-ordinates:

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right]\ . \tag{6.35}$$

6.13 Grad Div, Curl, ∇^2 in cylindrical polars

Here $(u, v, w) \to (r, \phi, z)$. The position vector is $\mathbf{r} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $h_r = |\partial\mathbf{r}/\partial r|$, etc.

$$\begin{aligned} \Rightarrow h_r &= \sqrt{(\cos^2\phi + sin^2\phi)} = 1, \\ h_\phi &= \sqrt{(r^2\sin^2\phi + r^2\cos^2\phi)} = r, \\ h_z &= 1 \end{aligned} \tag{6.36}$$

$$\begin{aligned} \Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \\ \text{div}\mathbf{a} &= \frac{1}{r}\left(\frac{\partial(ra_r)}{\partial r} + \frac{\partial a_\phi}{\partial\phi}\right) + \frac{\partial a_z}{\partial z} \\ \text{curl}\mathbf{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial(ra_\phi)}{\partial r} - \frac{\partial a_r}{\partial\phi}\right)\hat{\mathbf{k}} \\ \nabla^2 U &= \text{Tutorial Exercise} \end{aligned} \tag{6.37}$$

6.14 Grad Div, Curl, ∇^2 in spherical polars

Here $(u, v, w) \to (r, \theta, \phi)$. The position vector is $\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}$.

$$\begin{aligned} \Rightarrow h_r &= \sqrt{(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)} = 1 \\ h_\theta &= \sqrt{(r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta)} = r \\ h_\phi &= \sqrt{(r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta \end{aligned} \tag{6.38}$$

$$\begin{aligned} \Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}\hat{\boldsymbol{\phi}} \\ \text{div}\mathbf{a} &= \frac{1}{r^2}\frac{\partial(r^2a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\ \text{curl}\mathbf{a} &= \frac{\hat{\mathbf{r}}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \\ &\qquad \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right) \\ \nabla^2 U &= \text{Tutorial Exercise} \end{aligned} \tag{6.39}$$

♣ Examples

Q1 Find curl**a** in (i) Cartesians and (ii) Spherical polars when $\mathbf{a} = x(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$.

A1 (i) In Cartesians

$$\text{curl}\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\mathbf{j}} + y\hat{\mathbf{k}} \ . \tag{6.40}$$

(ii) In spherical polars, $x = r\sin\theta\cos\phi$ and $\mathbf{r} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$. So

$$\begin{aligned} \mathbf{a} &= r^2\sin\theta\cos\phi\hat{\mathbf{r}} \qquad (6.41) \\ \Rightarrow a_r &= r^2\sin\theta\cos\phi; \qquad a_\theta = 0; \qquad a_\phi = 0 \ . \end{aligned}$$

Hence as

$$\text{curl}\mathbf{a} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right) \tag{6.42}$$

$$\begin{aligned} \text{curl}\mathbf{a} &= \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(r^2\sin\theta\cos\phi)\right) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(-\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)\right) \qquad (6.43) \\ &= \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}(-r^2\sin\theta\sin\phi) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(-r^2\cos\theta\cos\phi)\right) \\ &= \hat{\boldsymbol{\theta}}(-r\sin\phi) + \hat{\boldsymbol{\phi}}(-r\cos\theta\cos\phi) \end{aligned}$$

Checking: these two results should be the same, but to check we need expressions for $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{i}}$ etc.

Remember that we can work out the unit vectors $\hat{\mathbf{r}}$ and so on in terms of $\hat{\mathbf{i}}$ etc using

$$\hat{\mathbf{r}} = \frac{1}{h_1}\frac{\partial\mathbf{r}}{dr}; \qquad \hat{\boldsymbol{\theta}} = \frac{1}{h_2}\frac{\partial\mathbf{r}}{d\theta}; \qquad \hat{\boldsymbol{\phi}} = \frac{1}{h_3}\frac{\partial\mathbf{r}}{d\phi} \quad \text{where } \mathbf{r} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}} \ . \tag{6.44}$$

Grinding through we find

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} = [R] \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \tag{6.45}$$

Don't be shocked to see a rotation matrix $[R]$: we are after all rotating one right-handed orthogonal coord system into another.

So the result in spherical polars is

$$\begin{aligned}\text{curl}\mathbf{a} &= (\cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}})(-r\sin\phi) + \\ &\qquad (-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}})(-r\cos\theta\cos\phi) \qquad (6.46)\\ &= -r\cos\theta\hat{\mathbf{j}} + r\sin\theta\sin\phi\hat{\mathbf{k}} \\ &= -z\hat{\mathbf{j}} + y\hat{\mathbf{k}}\end{aligned}$$

which is exactly the result in Cartesians.

Q2 Find the divergence of the vector field $\mathbf{a} = r\mathbf{c}$ where $\mathbf{c}$ is a constant vector (i) using Cartesian coordinates and (ii) using Spherical Polar coordinates.

A2 (i) Using Cartesian coords:

$$\begin{aligned}\text{div}\mathbf{a} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}c_x + \ldots \qquad (6.47)\\ &= x.(x^2+y^2+z^2)^{-1/2}c_x + \ldots \\ &= \frac{1}{r}\mathbf{r}\cdot\mathbf{c} \quad .\end{aligned}$$

(ii) Using Spherical polars

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}} \qquad (6.48)$$

and our first task is to find a_r and so on. We can't do this by inspection, and finding their values requires more work than you might think! Recall

$$\begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{bmatrix} = [R]\begin{bmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{bmatrix} \qquad (6.49)$$

Now the point is the same point in space whatever the coordinate system, so

$$a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}} \qquad (6.50)$$

and using the inner product

$$\begin{aligned}
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \\
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top [R]^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} &= [R] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}
\end{aligned} \tag{6.51}$$

For our particular problem, $a_x = rc_x$, etc, where c_x is a constant, so now we can write down

$$\begin{aligned}
a_r &= r(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z) \\
a_\theta &= r(\cos\theta\cos\phi c_x + \cos\theta\sin\phi c_y - \sin\theta c_z) \\
a_\phi &= r(-\sin\phi c_x + \cos\phi c_y)
\end{aligned} \tag{6.52}$$

Now all we need to do is to bash out

$$\mathrm{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \tag{6.53}$$

In glorious detail this is

$$\begin{aligned}
\mathrm{div}\mathbf{a} = {} & 3\left(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z\right) + \\
& \frac{1}{\sin\theta}\left((\cos^2\theta - \sin^2\theta)(\cos\phi c_x + \sin\phi c_y) - 2\sin\theta\cos\theta c_z\right) + \\
& \frac{1}{\sin\theta}\left(-\cos\phi c_x - \sin\phi c_y\right)
\end{aligned} \tag{6.54}$$

A bit more bashing and you'll find

$$
\begin{aligned}
\text{div}\mathbf{a} &= \sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z \\
&= \hat{\mathbf{r}}\cdot\mathbf{c}
\end{aligned}
\quad (6.55)
$$

This is EXACTLY what you worked out before of course.

Take home messages from these examples:

- Just as physical vectors are independent of their coordinate systems, so are differential operators.
- Don't forget about the vector geometry you did in the 1st year. Rotation matrices are useful!
- Spherical polars were NOT a good coordinate system in which to think about this problem. Let the symmetry guide you.

Chapter 27 – Magnetic Field and Magnetic Forces

- Magnetism
- Magnetic Field
- Magnetic Field Lines and Magnetic Flux
- Motion of Charged Particles in a Magnetic Field
- Applications of Motion of Charged Particles
- Magnetic Force on a Current-Carrying Conductor
- Force and Torque on a Current Loop

1) A moving charge or collection of moving charges (e.g. electric current) produces a magnetic field. (Chap. 28).

2) A second current or charge responds to the magnetic field and experiences a magnetic force. (Chap. 27).

1. Magnetism

Permanent magnets: exert forces on each other as well as on unmagnetized Fe pieces.

- The needle of a compass is a piece of magnetized Fe.

- If a bar-shaped permanent magnet is free to rotate, one end points north (north pole of magnet).

- An object that contains Fe is not by itself magnetized, it can be attracted by either the north or south pole of permanent magnet.

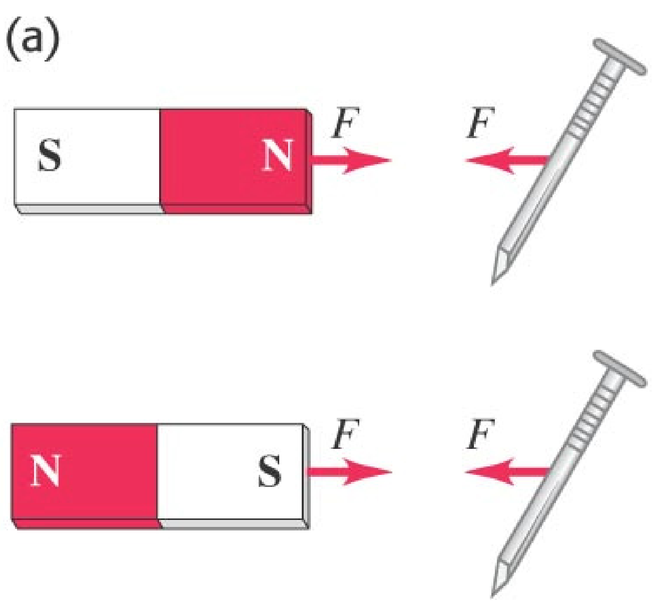
- A bar magnet sets up a magnetic field in the space around it and a second body responds to that field. A compass needle tends to align with the magnetic field at the needle's position.

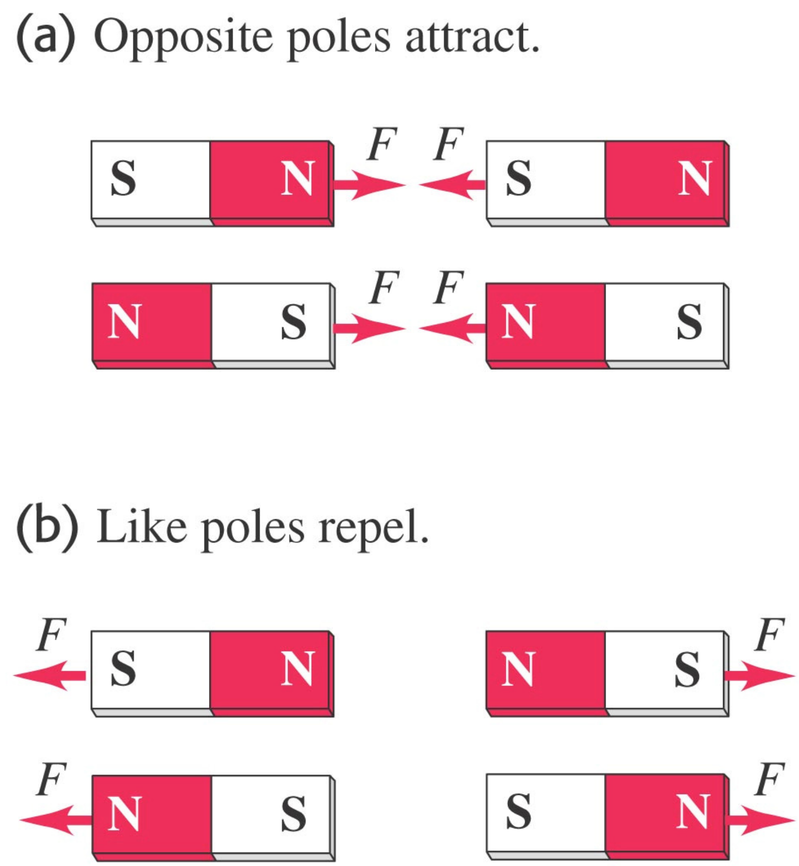
1. Magnetism

- Magnets exert forces on each other just like charges. You can draw magnetic field lines just like you drew electric field lines.

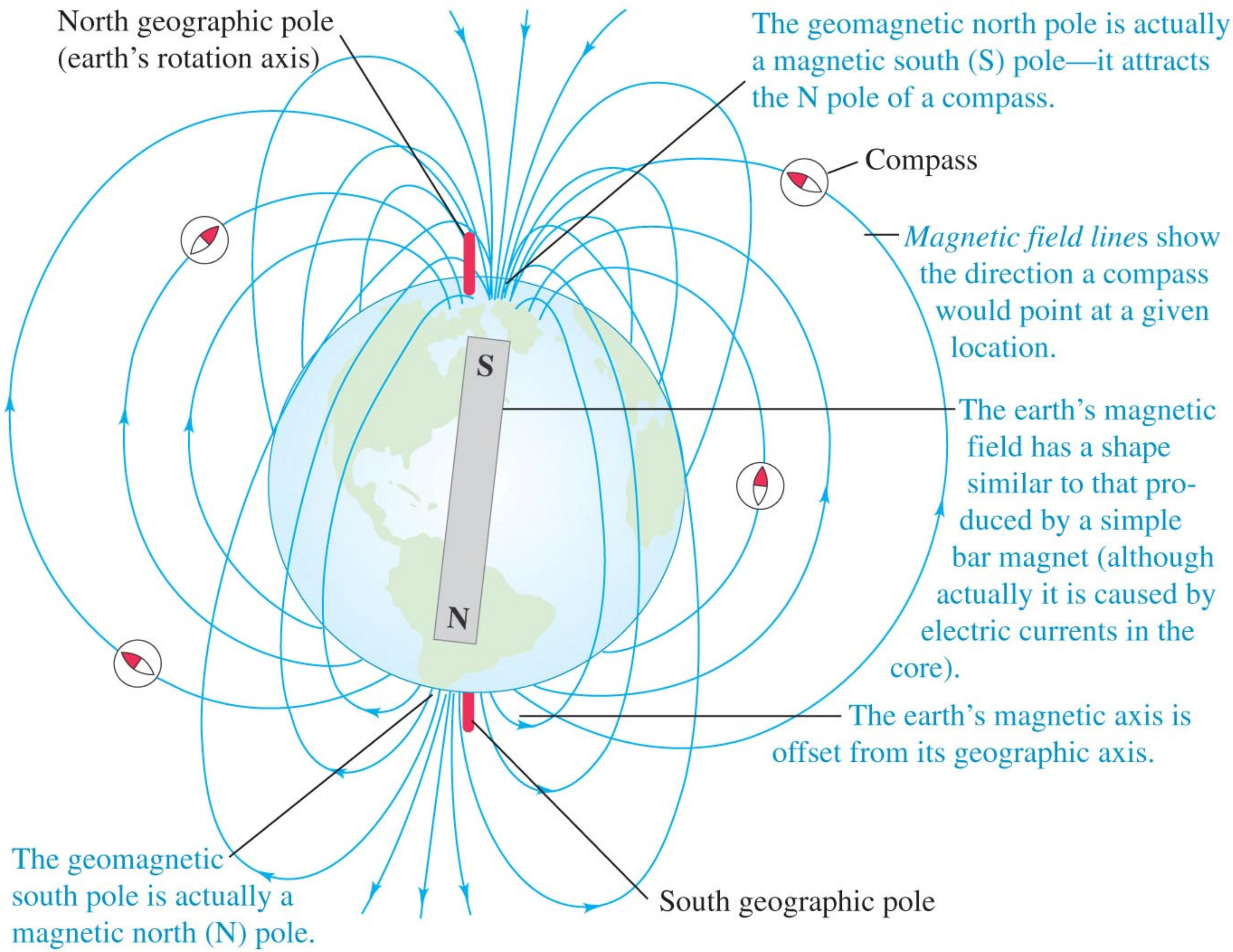
- Magnetic north and south pole's behavior is not unlike electric charges. For magnets, like poles repel and opposite poles attract.

- A permanent magnet will attract a metal like iron with either the north or south pole.





Magnetic poles about our planet



Magnetic declination / magnetic variation: the Earth's magnetic axis is not parallel to its geographic axis (axis of rotation) → a compass reading deviates from geographic north.

Magnetic inclination: the magnetic field is not horizontal at most of earth's surface, its angle up or down. The magnetic field is vertical at magnetic poles.

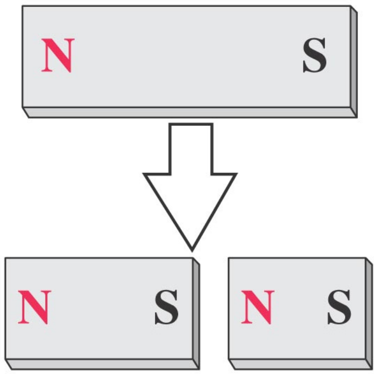
Magnetic Poles versus Electric Charge

- We observed monopoles in electricity. A (+) or (-) alone was stable, and field lines could be drawn around it.

- Magnets cannot exist as monopoles. If you break a bar magnet between N and S poles, you get two smaller magnets, each with its own N and S pole.

In contrast to electric charges, magnetic poles always come in pairs and can't be isolated.

Breaking a magnet in two ...



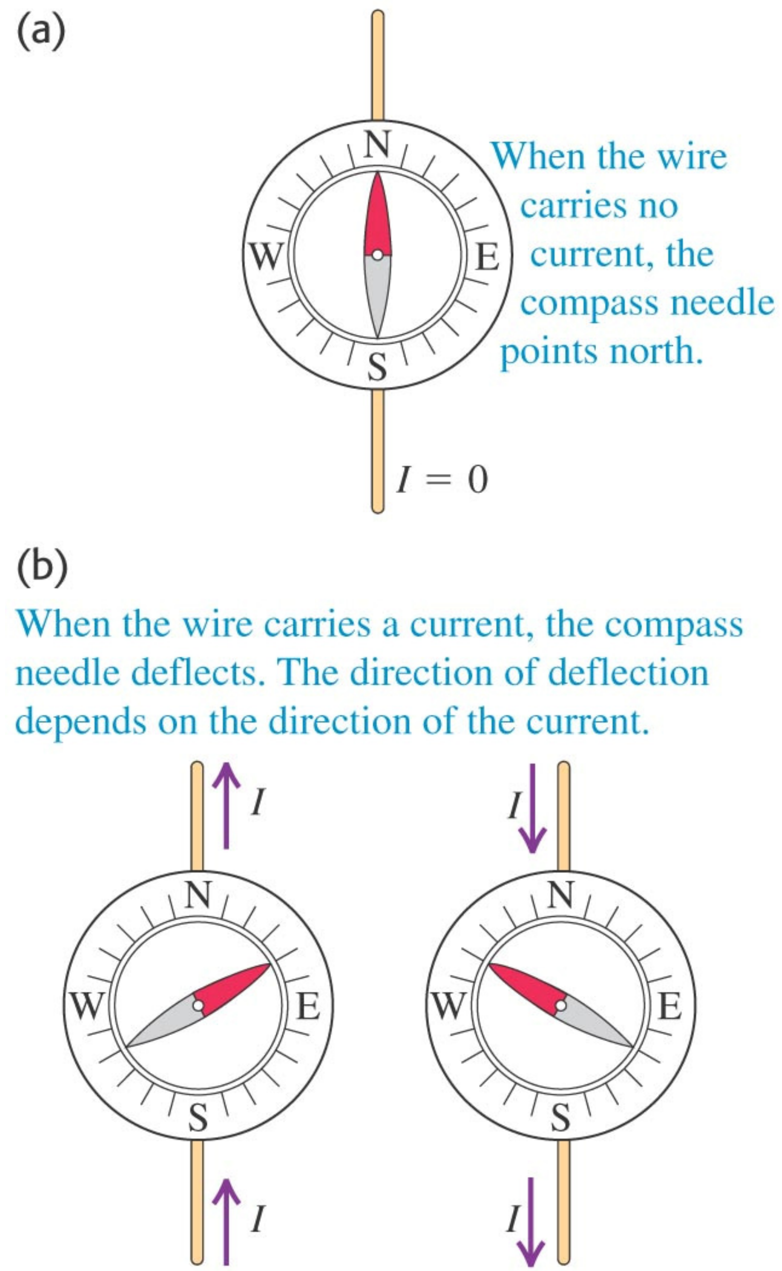
... yields two magnets, not two isolated poles.

-In 1820, Oersted ran experiments with conducting wires run near a sensitive compass. The orientation of the wire and the direction of the flow both moved the compass needle.

- Ampere / Faraday / Henry → moving a magnet near a conducting loop can induce a current.

- The magnetic forces between two bodies are due to the interaction between moving electrons in the atoms.

- Inside a magnetized body (permanent magnet) there is a coordinated motion of certain atomic electrons. Not true for unmagnetized objects.



2. Magnetic Field

Electric field:

1) A distribution of electric charge at rest creates an electric field E in the surrounding space.

2) The electric field exerts a force $\vec{F}_E = q\,\vec{E}$ on any other charges in presence of that field.

Magnetic field:

1) A moving charge or current creates a magnetic field in the surrounding space (in addition to $\vec{E}$).

2) The magnetic field exerts a force $\vec{F}_m$ on any other moving charge or current present in that field.

- The magnetic field is a vector field → vector quantity associated with each point in space.

$$F_m = |q| v_\perp B = |q| v\ B \sin\varphi$$

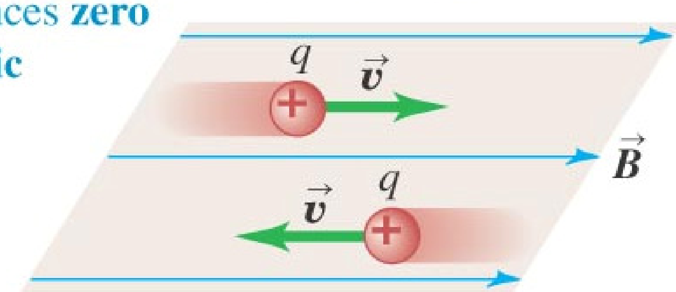
$$\vec{F}_m = q\vec{v} \times \vec{B}$$

- $\vec{F}_m$ is always perpendicular to $\vec{B}$ and $\vec{v}$.

2. Magnetic Field

- The moving charge interacts with the fixed magnet. The force between them is at a maximum when the velocity of the charge is perpendicular to the magnetic field.

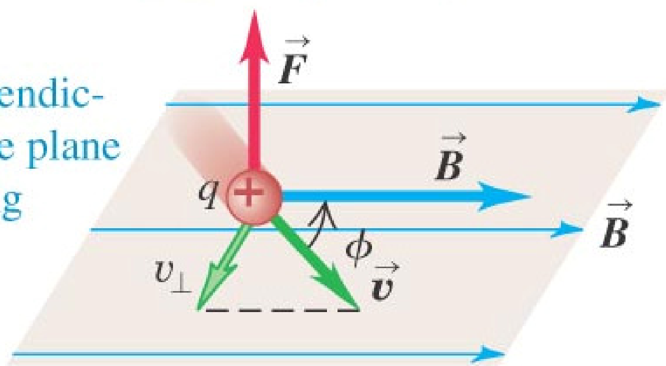
A charge moving **parallel** to a magnetic field experiences **zero magnetic force.**



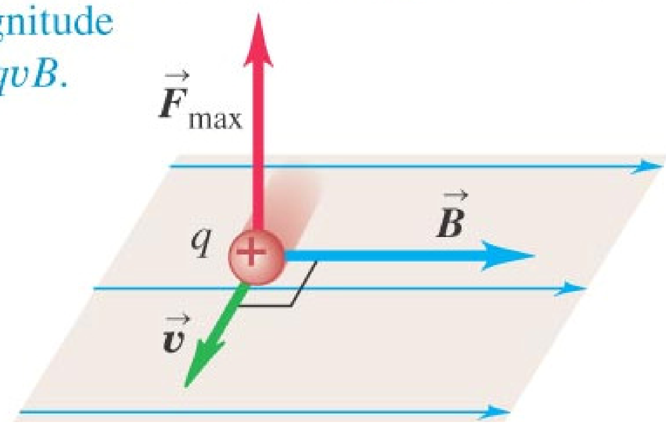
Interaction of magnetic force and charge

A charge moving at an angle ϕ to a magnetic field experiences a magnetic force with magnitude $F = |q|v_{\perp}B = |q|vB\sin\phi$.

$\vec{F}$ is perpendicular to the plane containing $\vec{v}$ and $\vec{B}$.



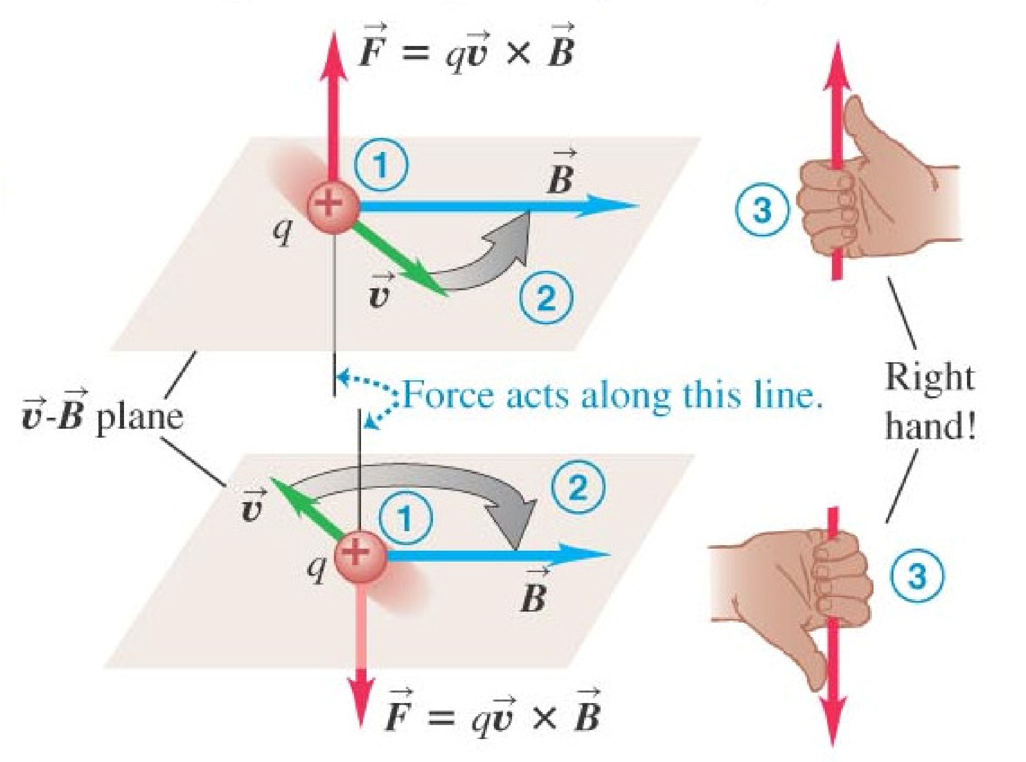
A charge moving **perpendicular** to a magnetic field experiences a maximal magnetic force with magnitude $F_{max} = qvB$.

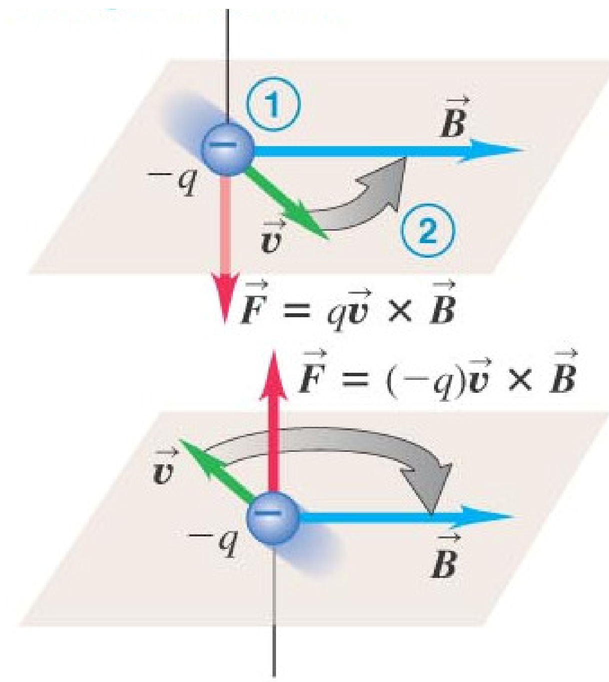


Right Hand Rule

Positive charge moving in magnetic field →direction of force follows right hand rule

Negative charge → F direction contrary to right hand rule.



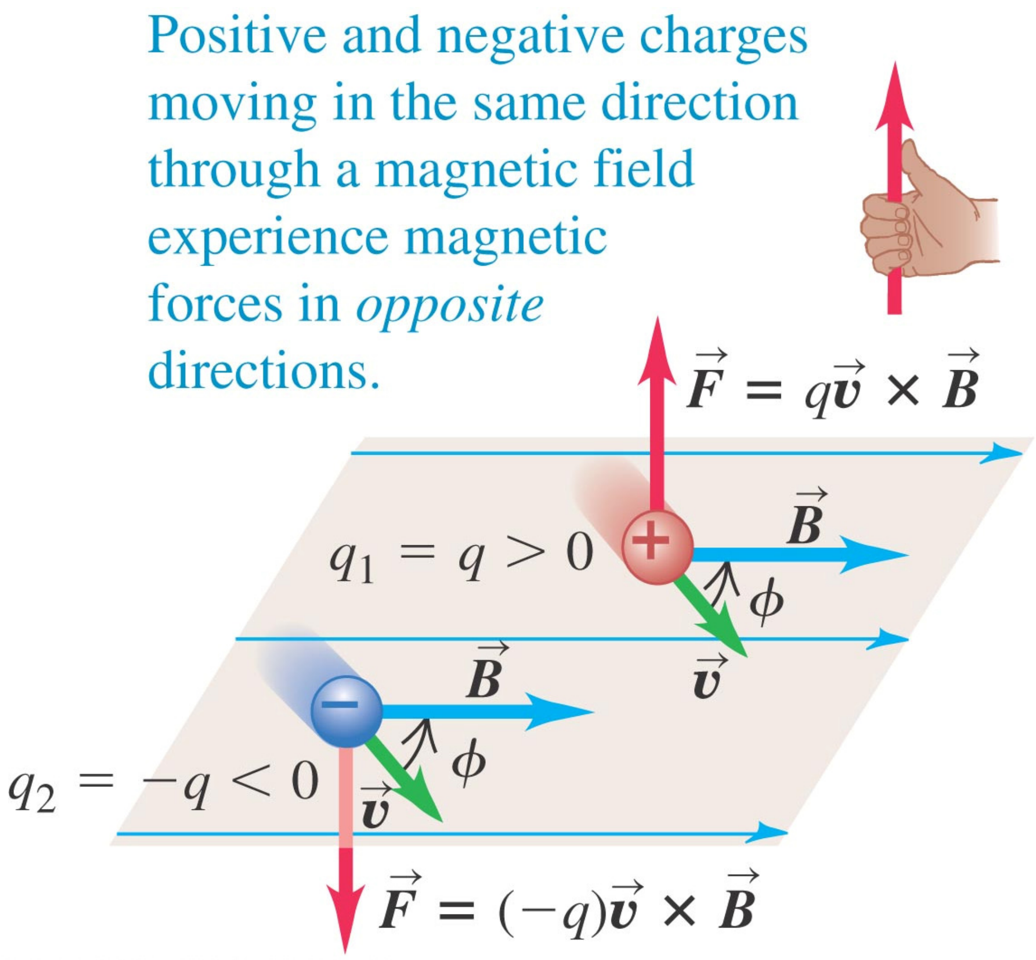


$$F = |q| v B_{\perp}$$

Units: 1 Tesla = 1 N s / C m = 1 N/A m

1 Gauss = 10^{-4} T

Right Hand Rule



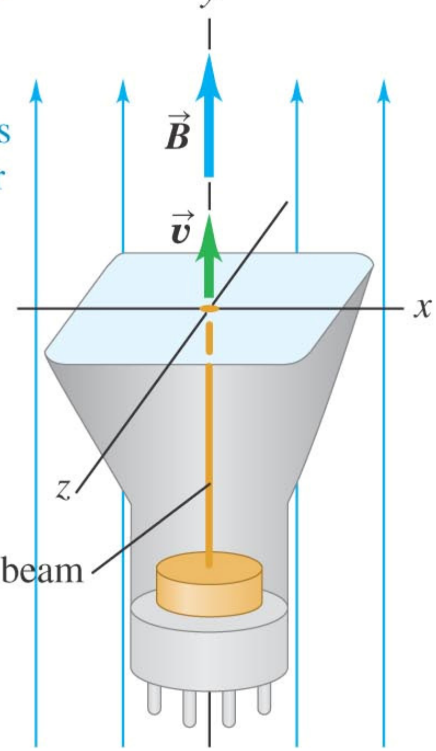
If charged particle moves in region where both, E and B are present:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

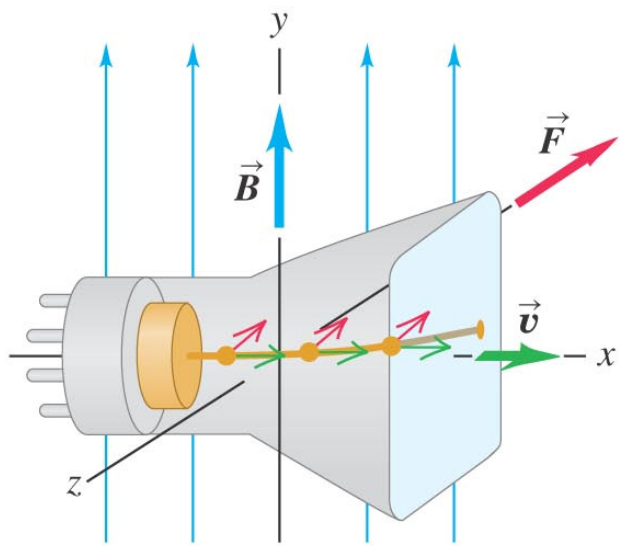
Measuring Magnetic Fields with Test Charges

Ex: electron beam in a cathode X-ray tube.

- In general, if a magnetic field (B) is present, the electron beam is deflected. However this is not true if the beam is // to B ($\varphi = 0, \pi$ → F=0 → no deflection).



(a) If the tube axis is parallel to the y-axis, the beam is undeflected, so $\vec{B}$ is in either the $+y$- or the $-y$-direction.



(b) If the tube axis is parallel to the x-axis, the beam is deflected in the $-z$-direction, so $\vec{B}$ is in the $+y$-direction.

Electron q< 0 → F has contrary direction to right hand rule

No deflection → $\vec{F} = 0$ → $\vec{v}$ // $\vec{B}$

Deflection → $\vec{F} \neq 0$ → $\vec{F} \perp \vec{v}, \vec{B}$

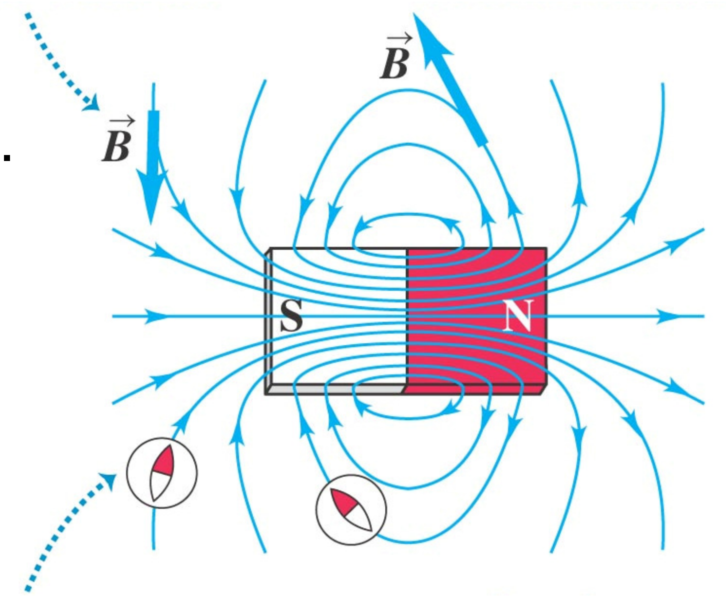
3. Magnetic Field Lines and Magnetic Flux

- Magnetic field lines may be traced from N toward S (analogous to the electric field lines).

- At each point they are tangent to magnetic field vector.

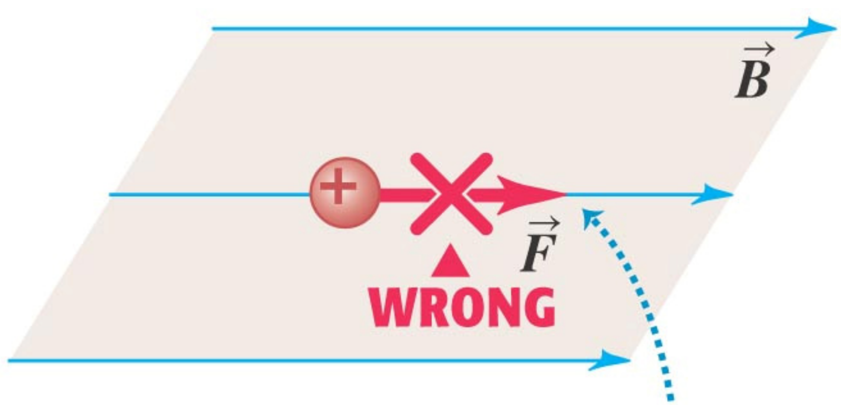
- The more densely packed the field lines, the stronger the field at a point.

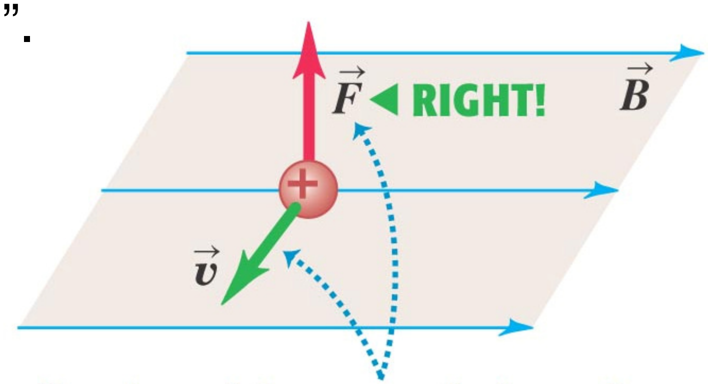
- Field lines never intersect.



- The field lines point in the same direction as a compass (from N toward S).

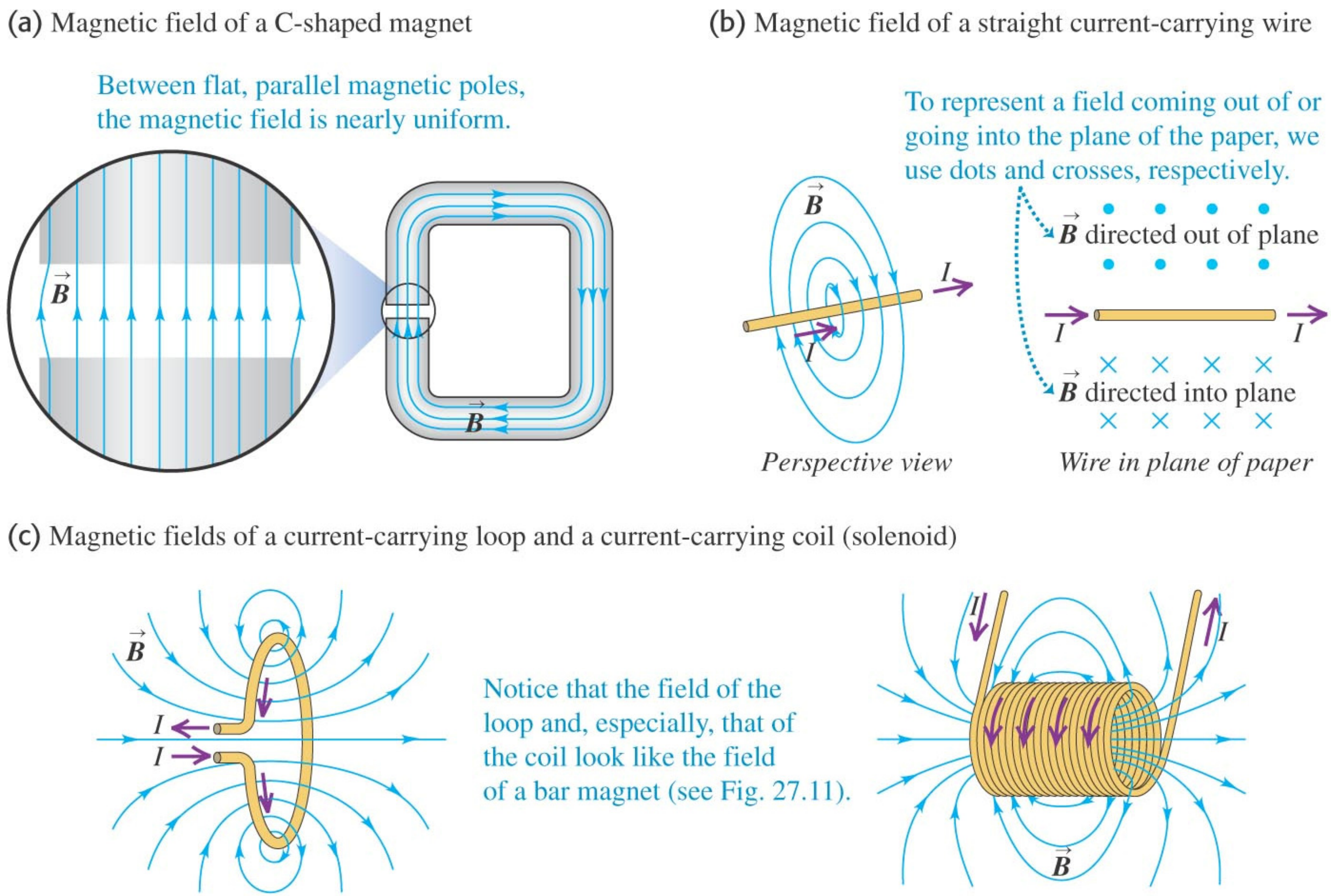
- Magnetic field lines are not "lines of force".





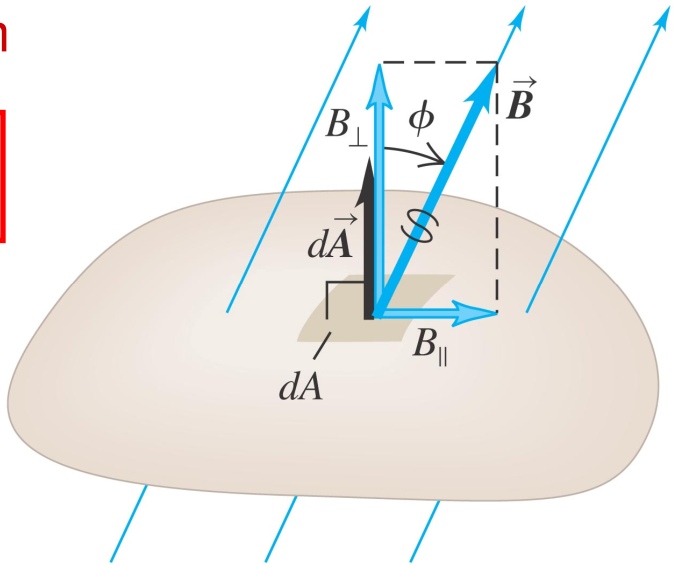
The direction of the magnetic force depends on the velocity $\vec{v}$, as expressed by the magnetic force law $\vec{F} = q\vec{v} \times \vec{B}$.

- Magnetic field lines have no ends → they continue through the interior of the magnet.



Magnetic Flux and Gauss's Law for Magnetism

$$\Phi_B = \int B_\perp dA = \int B\cos\varphi \cdot dA = \int \vec{B}\cdot d\vec{A}$$



- Magnetic flux is a scalar quantity.

- If $\vec{B}$ is uniform: $$\Phi_B = B_\perp A = BA\cos\varphi$$

Units: 1 Weber (1 Wb = 1 T m^2 = 1 N m / A)

- Difference with respect to electric flux → the total magnetic flux through a closed surface is always zero. This is because there is no isolated magnetic charge ("monopole") that can be enclosed by the Gaussian surface.

$$\Phi_B = \oint \vec{B}\cdot d\vec{A} = 0$$

$$B = \frac{d\Phi_B}{dA_\perp}$$

- The magnetic field is equal to the flux per unit area across an area at right angles to the magnetic field = magnetic flux density.

4. Motion of Charged Particles in a Magnetic Field

- Magnetic force perpendicular to $\vec{v}$ → it cannot change the magnitude of the velocity, only its direction.

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

- $\vec{F}$ does not have a component parallel to particle's motion → cannot do work.

- Motion of a charged particle under the action of a magnetic field alone is always motion with constant speed.

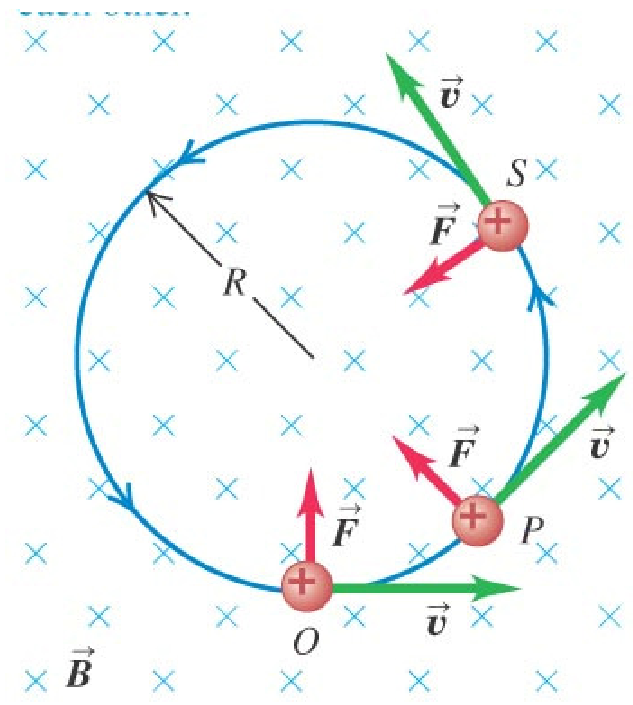
- Magnitudes of F and v are constant (v perp. B) → uniform circular motion.

$$F = |q| \cdot v \cdot B = m\frac{v^2}{R}$$

Radius of circular orbit in magnetic field:

$$R = \frac{mv}{|q|B}$$

+ particle → counter-clockwise rotation.
- particle → clockwise rotation.

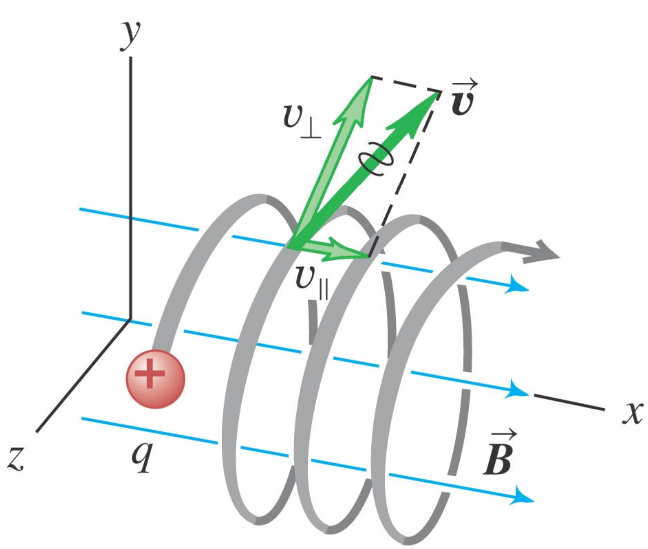


Angular speed: ω = v/R → $\omega = v\frac{|q|B}{mv} = \frac{|q|B}{m}$

Cyclotron frequency: f = ω/2π

- If v is not perpendicular to B → $v_{//}$ (parallel to B) constant because $F_{//} = 0$ → particle moves in a helix. (R same as before, with $v = v_{\perp}$).

This particle's motion has components both parallel ($v_{\parallel}$) and perpendicular ($v_{\perp}$) to the magnetic field, so it moves in a helical path.



A charged particle will move in a plane perpendicular to the magnetic field.

5. Applications of Motion of Charged Particles

Velocity selector

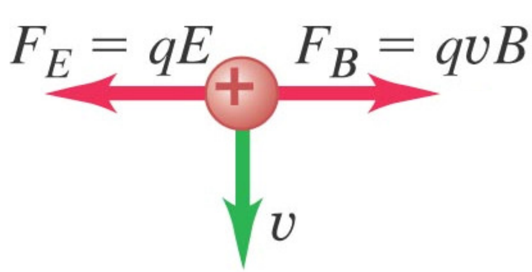
- Particles of a specific speed can be selected from the beam using an arrangement of E and B fields.

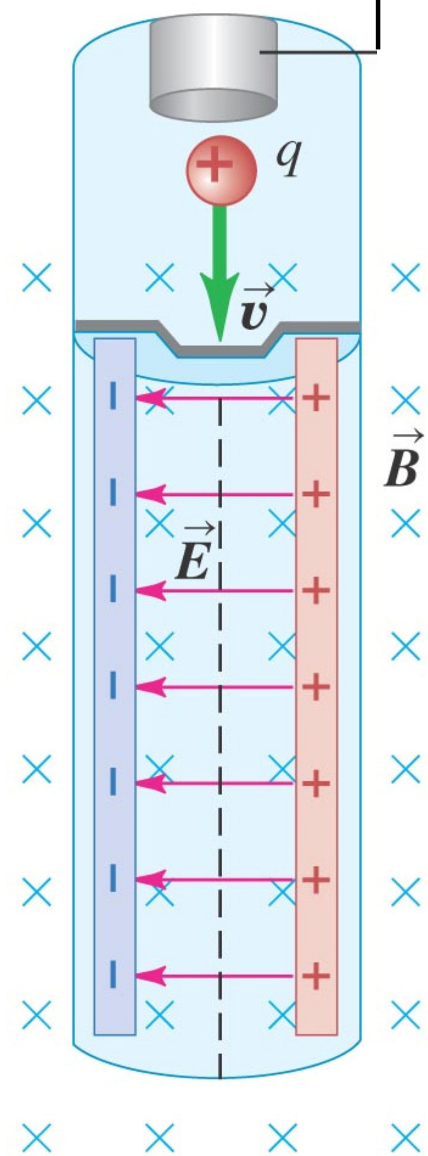
- F_m (magnetic) for + charge towards right (q v B).

- F_E (electric) for + charge to left (q E).

- $F_{net} = 0$ if $F_m = F_E$ → $-qE + q v B = 0$ → $v = E/B$

- Only particles with speed E/B can pass through without being deflected by the fields.





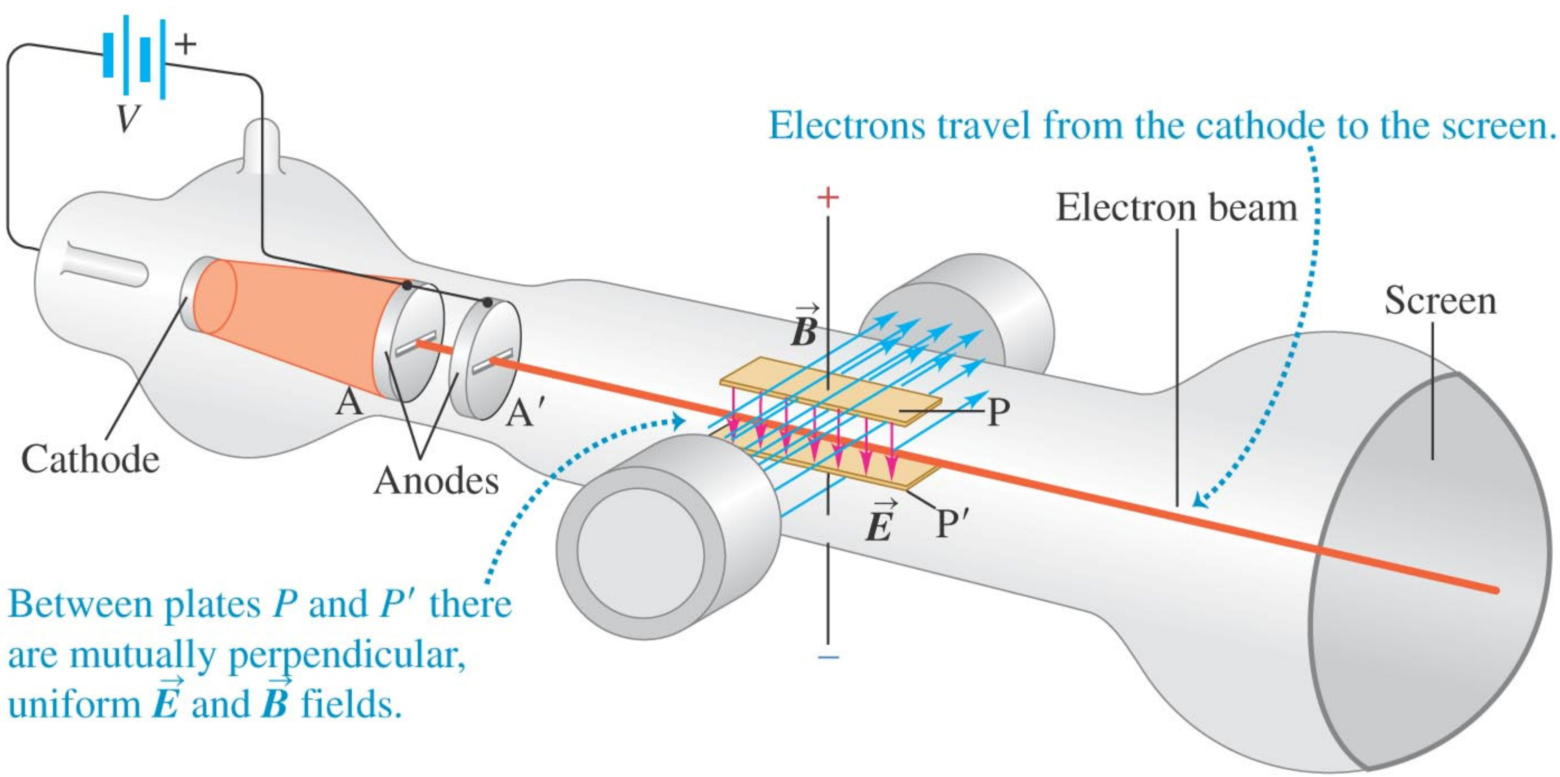
Thomson's *e/m* Experiment

$\Delta E = \Delta K + \Delta U = 0 \rightarrow 0.5\ m\ v^2 = U = e\ V$

$$v = \frac{E}{B} = \sqrt{\frac{2eV}{m}}$$

$$\boxed{\frac{e}{m} = \frac{E^2}{2VB^2}}$$

e/m does not depend on the cathode material or residual gas on tube → particles in the beam (electrons) are a common constituent of all matter.

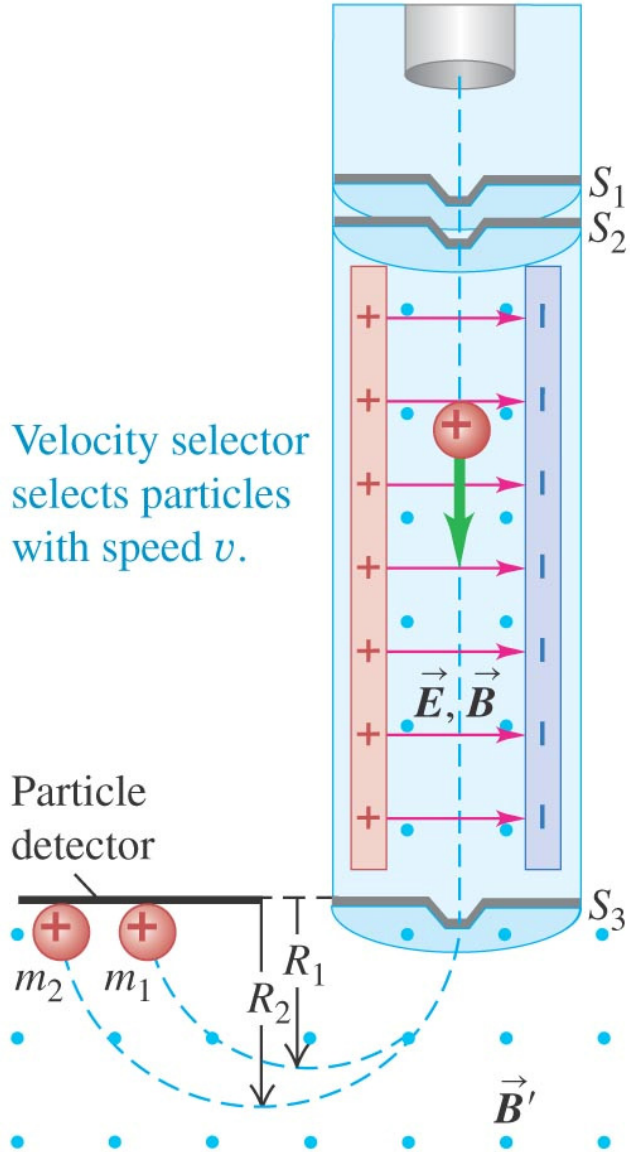


Mass Spectrometer

- Using the same concept as Thompson, Bainbridge was able to construct a device that would only allow one mass in flight to reach the detector.

- Velocity selector filters particles with v = E/B. After this, in the region of B' particles with $m_2 > m_1$ travel with radius ($R_2 > R_1$).

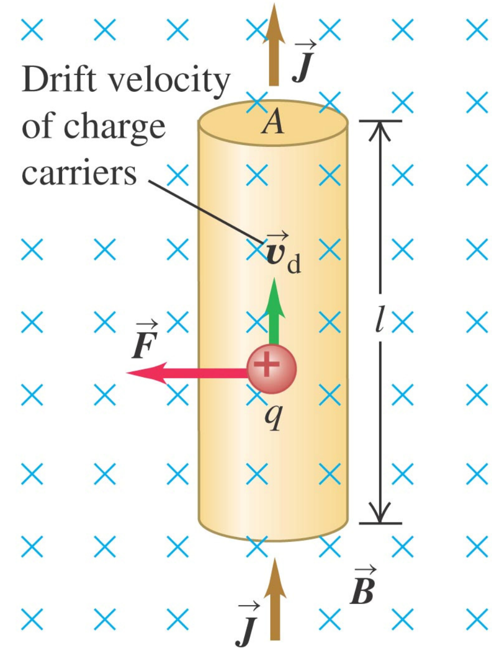
$$R = \frac{mv}{|q|B'}$$



Magnetic field separates particles by mass; the greater a particle's mass, the larger is the radius of its path.

6. Magnetic Force on a Current-Carrying Conductor

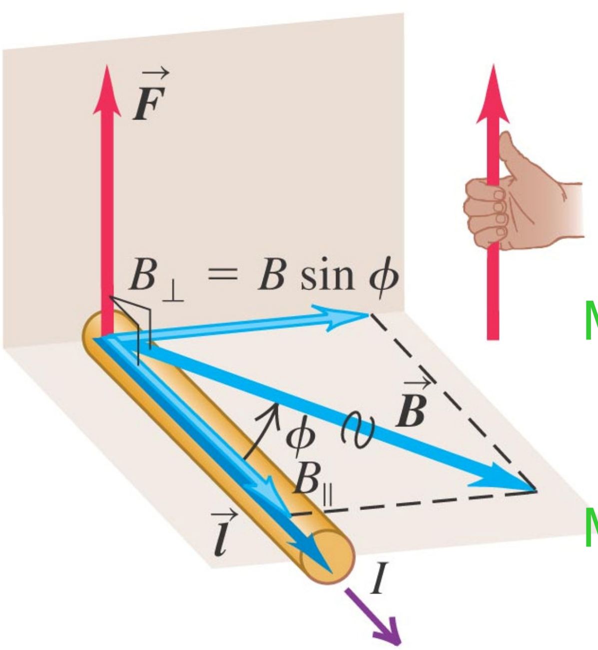
$$\vec{F}_m = q\vec{v}_d \times \vec{B} \qquad F_m = qv_d B \quad \text{Force on one charge}$$



- Total force: $F_m = (nAl)(qv_d B)$

n = number of charges per unit volume
A l = volume

$$F_m = (nqv_d)(A)(lB) = (JA)(lB) = IlB \qquad (\text{B} \perp \text{wire})$$



In general:

$$F = IlB_\perp = IlB\sin\varphi$$

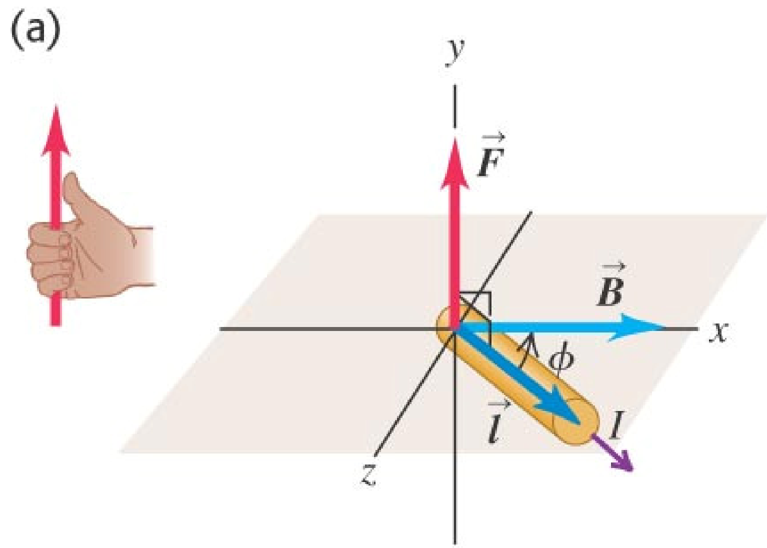
Magnetic force on a straight wire segment:

$$\boxed{\vec{F} = I\vec{l} \times \vec{B}}$$

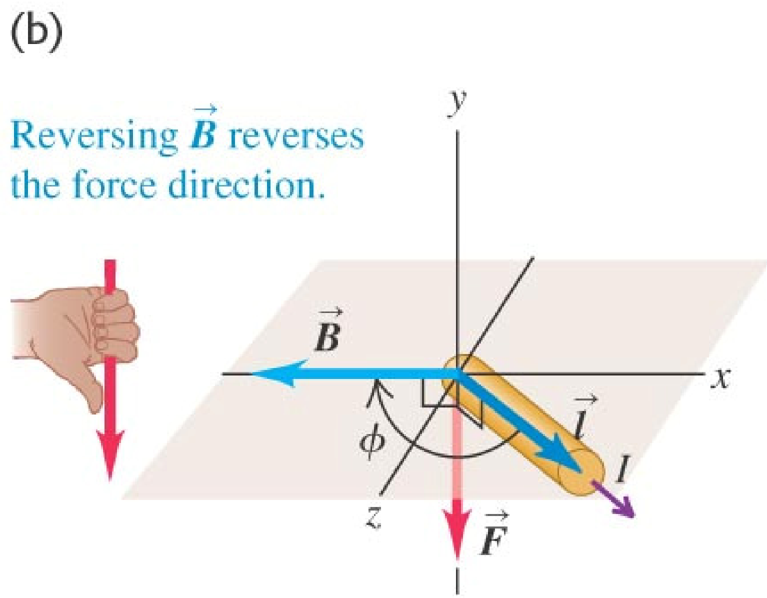
Magnetic force on an infinitesimal wire section:

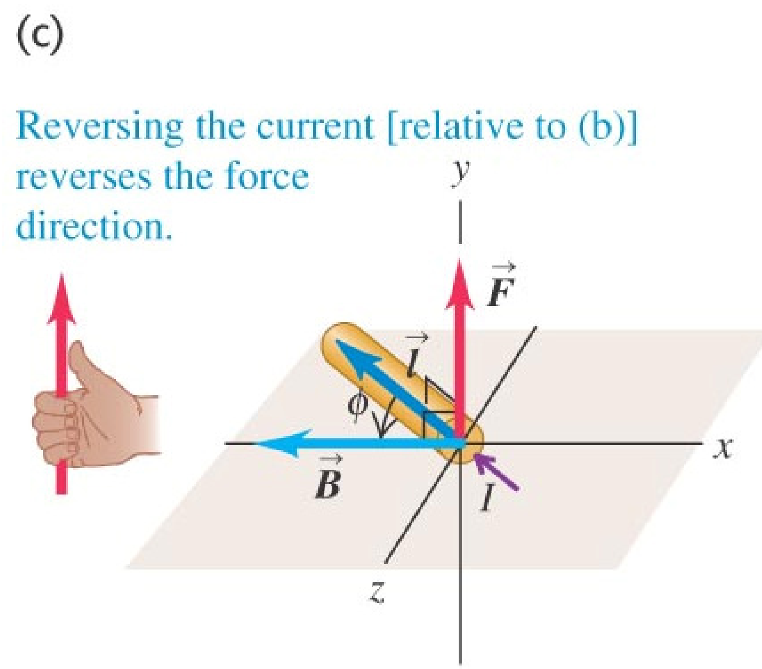
$$d\vec{F} = Id\vec{l} \times \vec{B}$$

- Current is not a vector. The direction of the current flow is given by $\vec{dl}$, not I. $\vec{dl}$ is tangent to the conductor.



$$\vec{F} = I\vec{l} \times \vec{B}$$





7. Force and Torque on a Current Loop

- The net force on a current loop in a uniform magnetic field is zero.

Right wire of length "a" → F = I a B (B ⊥ *l*)

Left wire of length "b" → F' = I b B sin (90º - φ) (B forms 90º-φ angle with *l*)

F' = I b B cos φ

F_{net} = F – F + F' - F' = 0

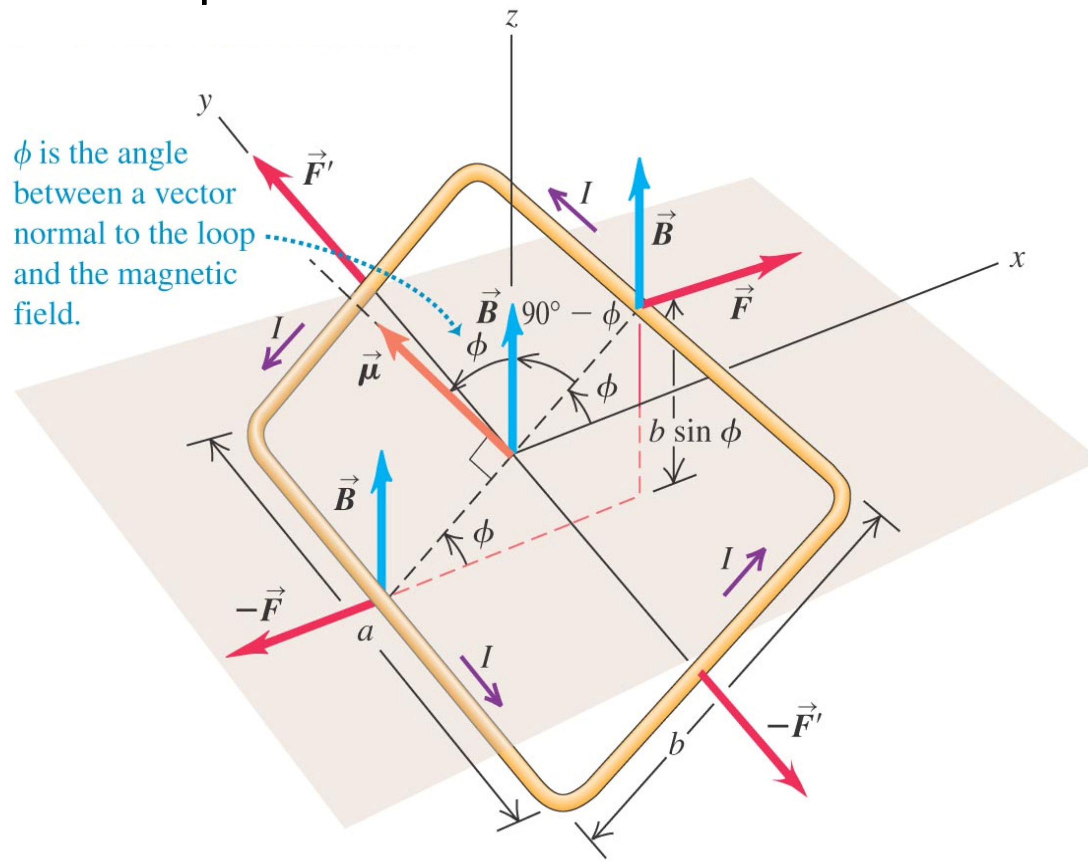
- Net torque ≠ 0 (general).

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = r \cdot F \sin\alpha = r_{\perp} F = r F_{\perp}$$

$$\tau_{F'} = r \cdot F \sin 0^{\circ} = 0$$

$$\tau_F = F(b/2)\sin\varphi$$

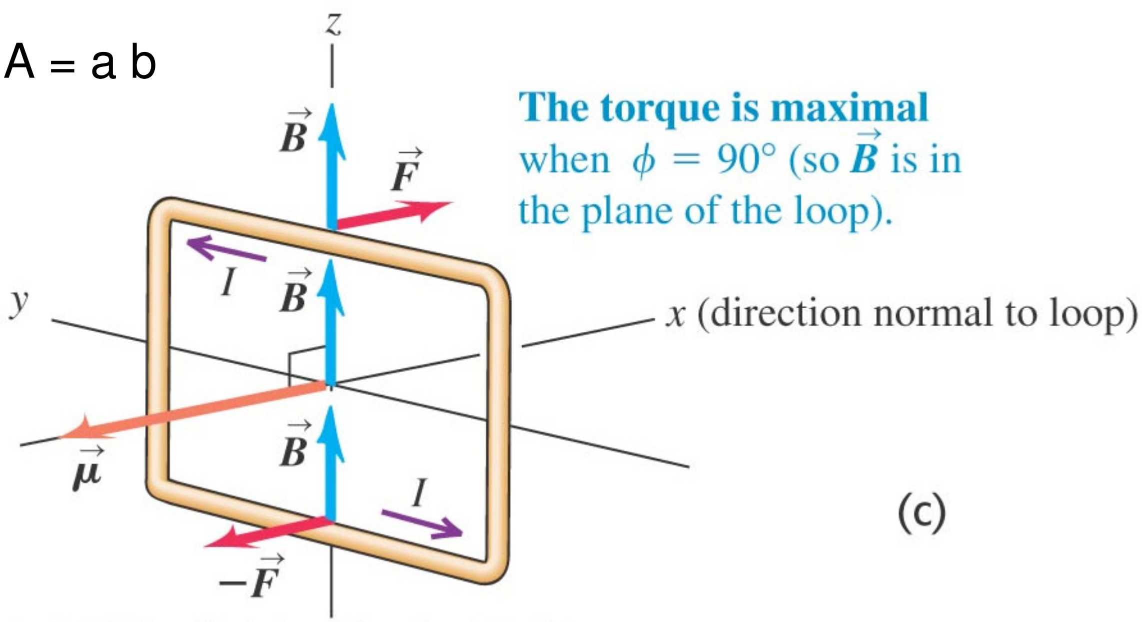


$$\tau_{total} = \tau_{F'} + \tau_{-F'} + \tau_F + \tau_{-F} = 0 + 0 + 2(b/2)F\sin\varphi$$

$$\boxed{\tau_{total} = (IBa)(b\sin\varphi) = IBA\sin\varphi}$$

Torque on a current loop

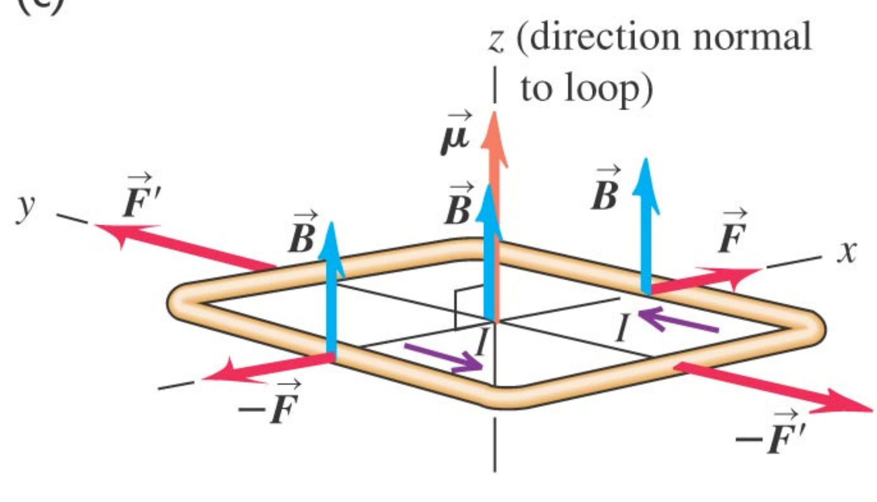
A = a b



φ is angle between a vector perpendicular to loop and $\vec{B}$

Torque is zero, φ = 0º

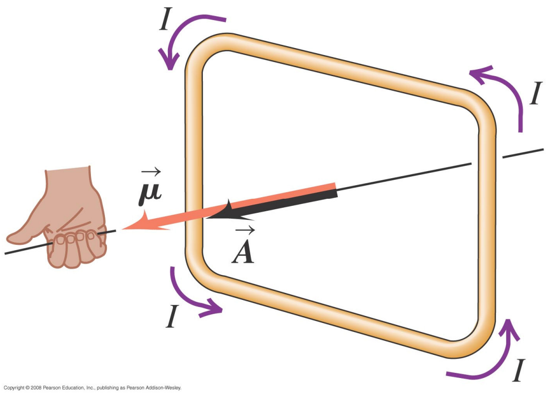
(c)



$$\tau_{total} = IBA\sin\varphi$$

Magnetic dipole moment: $\vec{\mu} = I\vec{A}$

Direction: perpendicular to plane of loop (direction of loop's vector area → right hand rule)



$$\tau_{total} = \mu B\sin\varphi$$

Magnetic torque: $\vec{\tau} = \vec{\mu}\times\vec{B}$

Potential Energy for a Magnetic Dipole:

$$U = -\vec{\mu}\cdot\vec{B} = -\mu B\cos\varphi$$

Electric dipole moment: $\overline{p} = q\vec{d}$

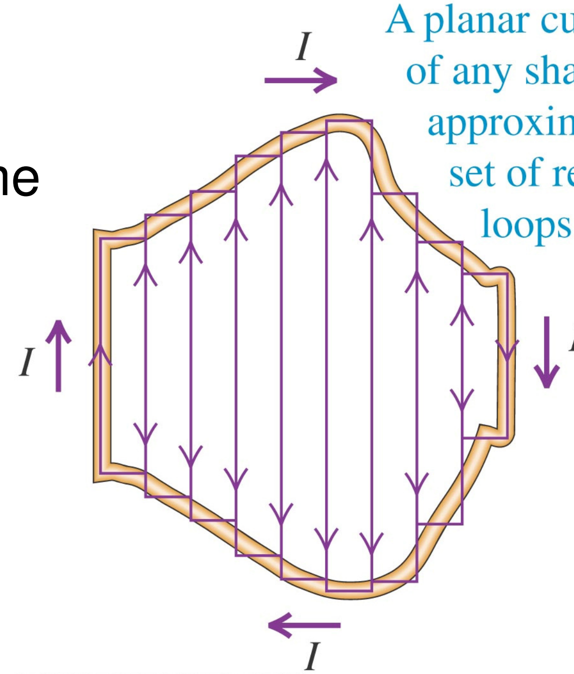
Electric torque: $\vec{\tau} = \vec{p}\times\vec{E}$

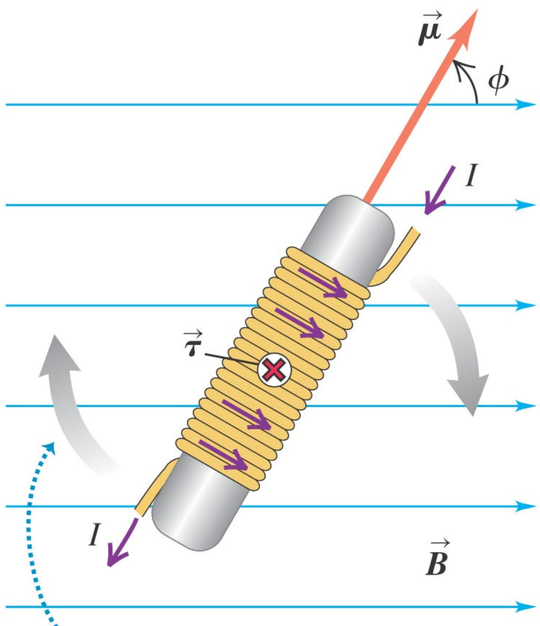
Potential Energy for an Electric Dipole:

$$U = -\vec{p}\cdot\vec{E}$$

Magnetic Torque: Loops and Coils

If these loops all carry equal current “I” in same clockwise sense, F and torque on the sides of two adjacent loops cancel, and only forces and torques around boundary ≠ 0.





The torque tends to make the solenoid rotate clockwise in the plane of the page, aligning magnetic moment $\vec{\mu}$ with field $\vec{B}$.

Solenoid

$$\tau = NIBA\sin\varphi$$

N = number of turns

φ is angle between axis of solenoid and B

Max. torque: solenoid axis ⊥ B.

Torque rotates solenoid to position where its axis is parallel to B.

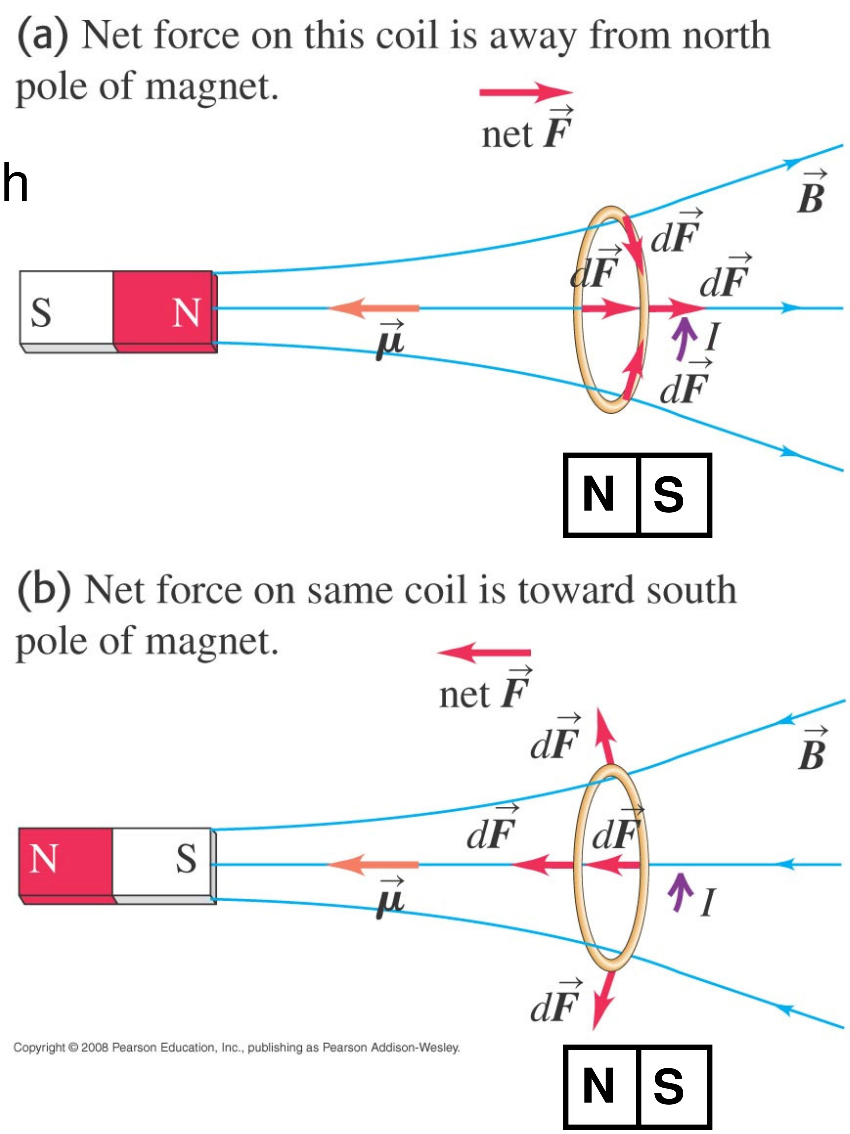
Magnetic Dipole in a Non-Uniform Magnetic Field

- Net force on a current loop in a non-uniform field is not zero.

$$d\vec{F} = I d\vec{l} \times \vec{B}$$

Radial force components cancel each Other → F_{net} to right.

If polarity of magnet changes → F_{net} to left.



Magnetic Dipole and How Magnets Work

A solenoid and a magnet orients themselves with axis parallel to field.

Electron analogy: “spinning ball of charge” → circulation of charge around spin axis similar to current loop → electron has net magnetic moment.

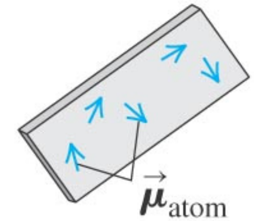
- In Fe atom, large number of electron magnetic moments align to each other → non-zero atomic magnetic moment.

- In unmagnetized Fe piece → no overall alignment of μ of atoms → total μ = 0.

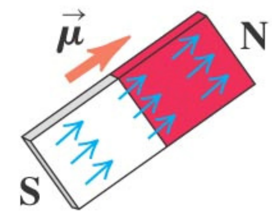
- Iron bar magnet → magnetic moments of many atoms are parallel → total μ ≠ 0.

- A bar magnet tends to align to B, so that line from S to N is in direction of B.

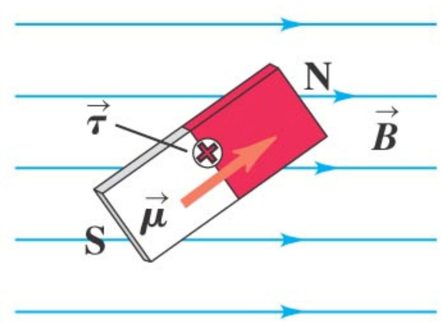
(a) Unmagnetized iron: magnetic moments are oriented randomly.



(b) In a bar magnet, the magnetic moments are aligned.



(c) A magnetic field creates a torque on the bar magnet that tends to align its dipole moment with the $\vec{B}$ field.



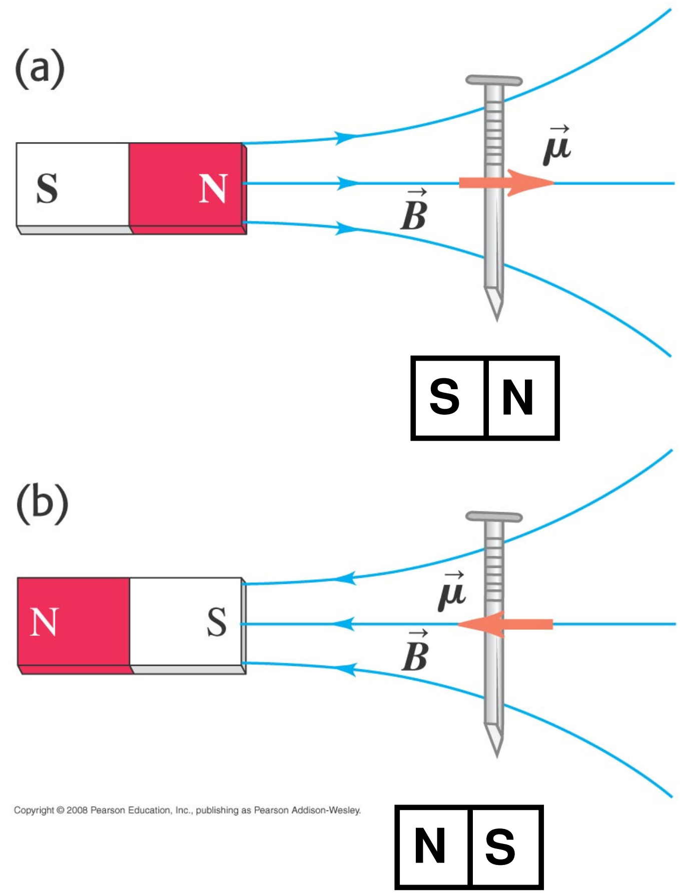
- South and North poles represent tail and head of magnet's dipole moment, μ.

How can a magnet attract an unmagnetized Fe object?

1) Atomic magnetic moments of Fe try to align to B of bar magnet → Fe acquires net magnetic dipole moment // B.

2) Non-Uniform B attracts magnetic dipole.

The magnetic dipole produced on nail is equivalent to current loop (I direction right hand rule) → net magnetic force on nail is attractive (a) or (b) → unmagnetized Fe object is attracted to either pole of magnet.



Chapter 29 – Electromagnetic Induction

- Induction Experiments
- Faraday’s Law
- Lenz’s Law
- Motional Electromotive Force
- Induced Electric Fields
- Eddy Currents
- Displacement Current and Maxwell’s Equations
- Superconductivity

- If the magnetic flux through a circuit changes, an emf and a current are induced.

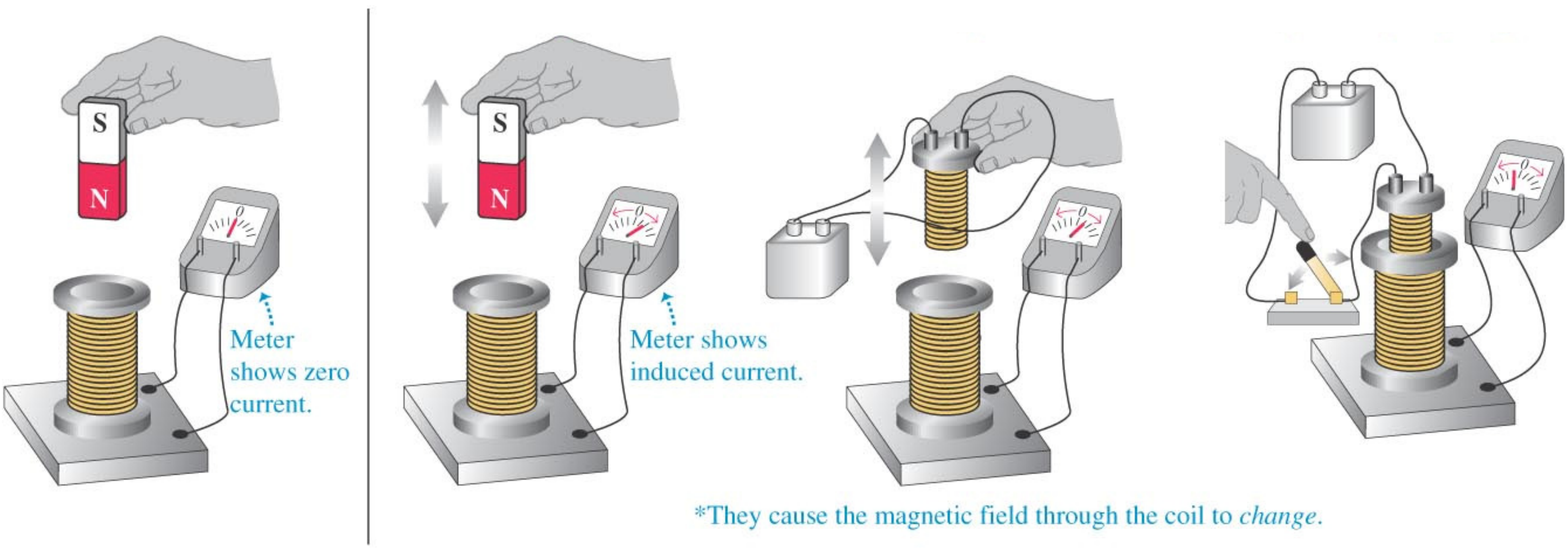
- A time-varying magnetic field can act as source of electric field.

- A time-varying electric field can act as source of magnetic field.

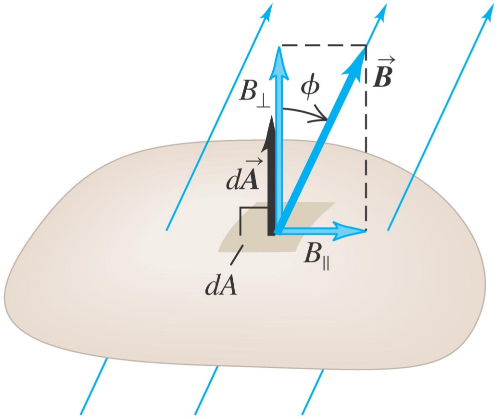
Maxwell

1. Induction Experiments (Faraday / Henry)

- An induced current (and emf) is generated when: (a) we move a magnet around a coil, (b) move a second coil toward/away another coil, (c) change the current in the second coil by opening/closing a switch.



- Magnetically induced emfs are always the result of the action of non-electrostatic forces. The electric fields caused by those forces are $\vec{E}_n$ (non-Coulomb, non conservative).

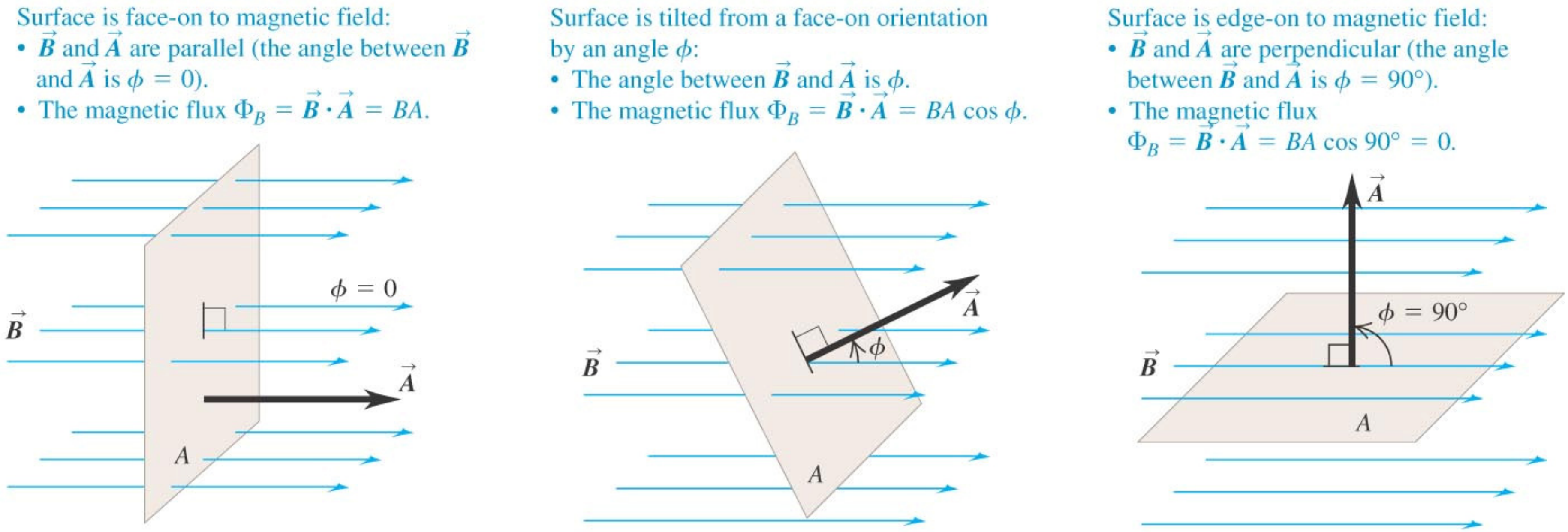


2. Faraday's Law

Magnetic flux:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B \cos\varphi \cdot dA$$

If $\vec{B}$ is uniform over a flat area $\vec{A}$: $\Phi_B = \vec{B} \cdot \vec{A} = B \cdot A \cdot \cos\varphi$



Faraday's Law of Induction:

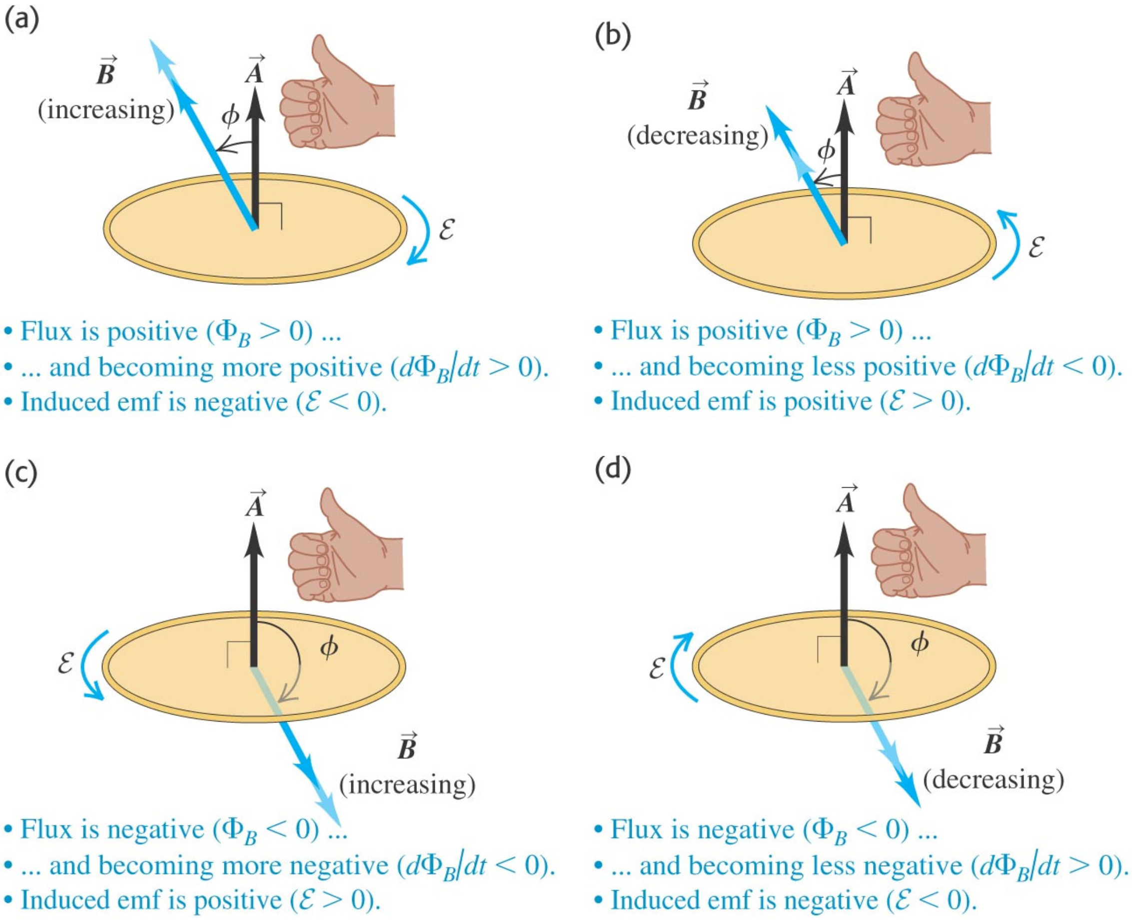
- *The induced emf in a closed loop equals the negative of the time rate of change of the magnetic flux through the loop.*

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

- Increasing flux → $\varepsilon < 0$;   Decreasing flux → $\varepsilon > 0$

- <u>Direction:</u> curl fingers of right hand around $\vec{A}$, if $\varepsilon > 0$ is in same direction of fingers (counter-clockwise), if $\varepsilon < 0$ contrary direction (clockwise).

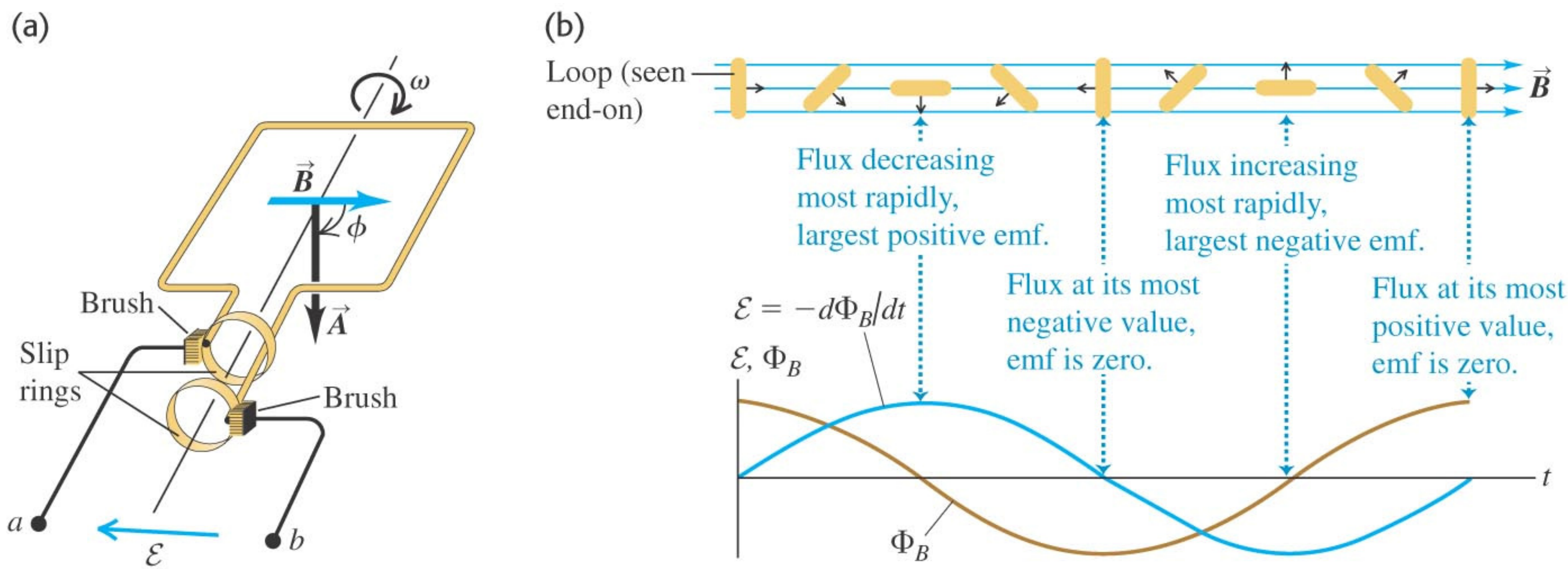
- Only a change in the flux through a circuit (not flux itself) can induce emf. If flux is constant → no induced emf.



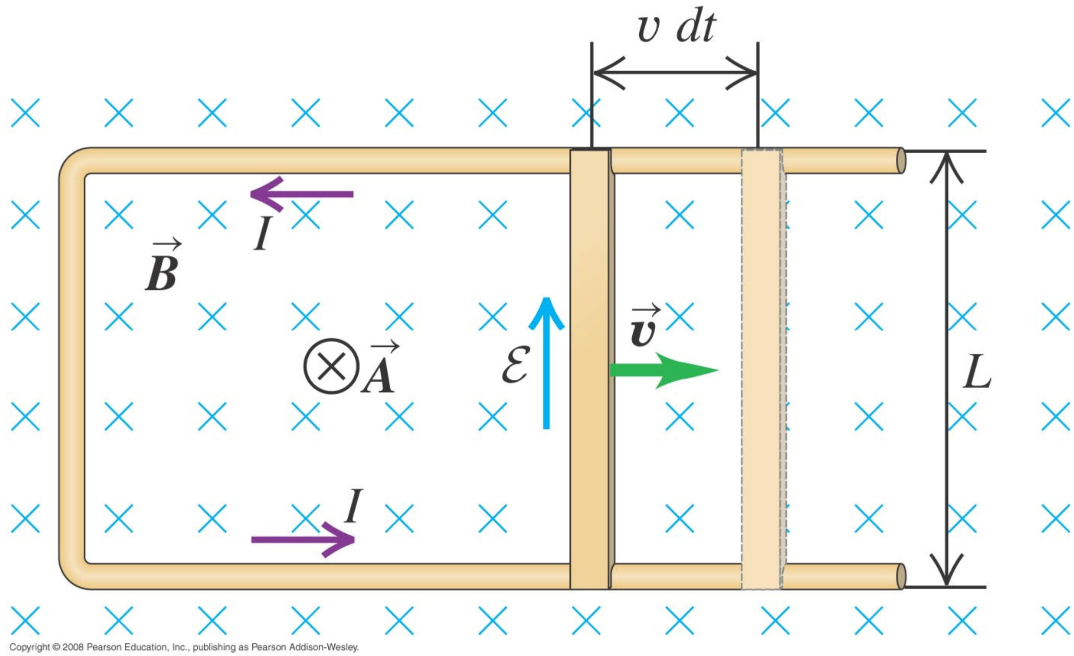
Coil: $$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$ N = number of turns

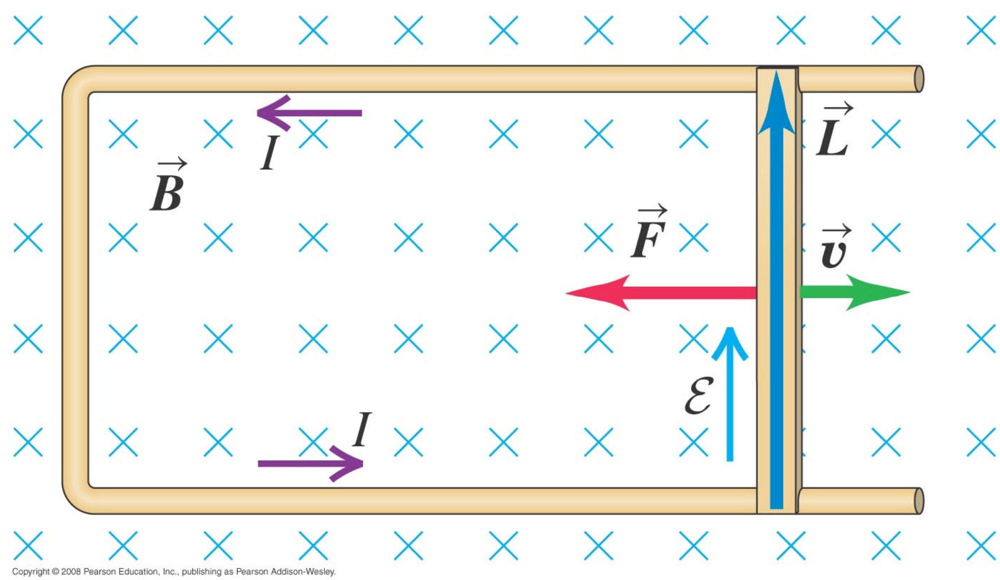
- If the loop is a conductor, an induced current results from emf. This current produces an additional magnetic field through loop. From right hand rule, that field is opposite in direction to the increasing field produced by electromagnet.

Ex: 29.4 - Generator I: a simple alternator



Exs: 29.6, 29.7 - Generator III: the slide wire generator



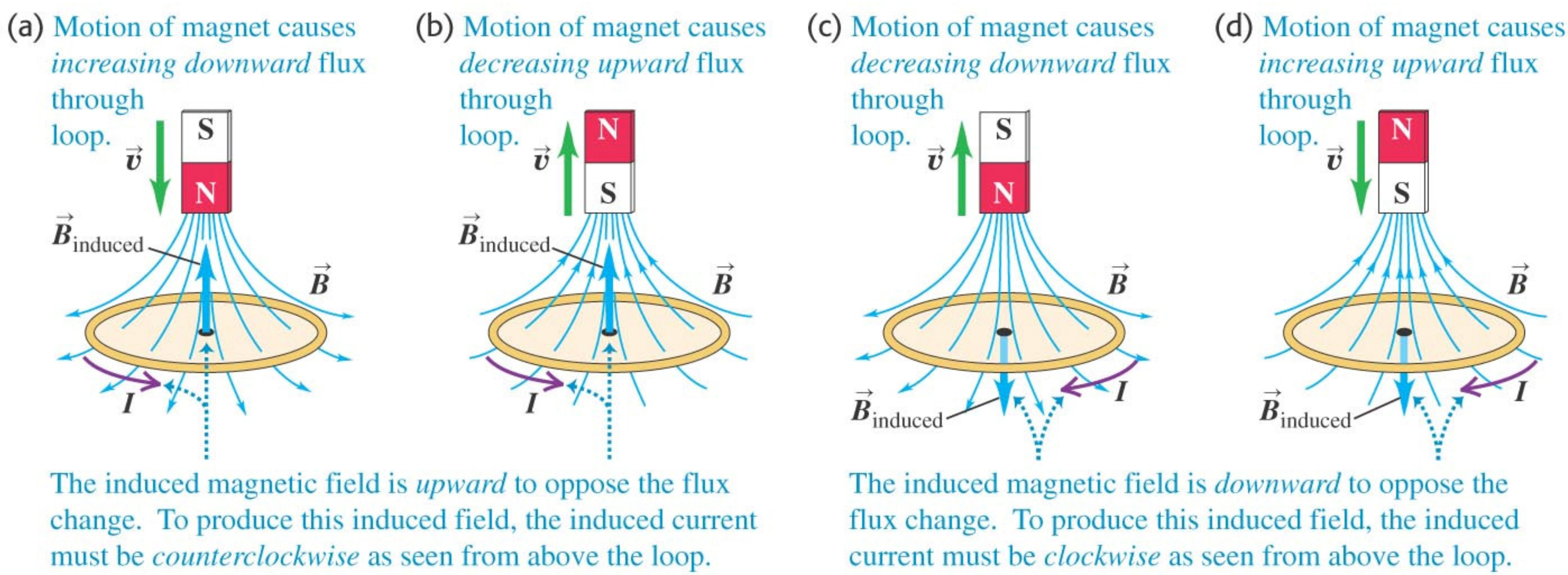


3. Lenz's Law

- Alternative method for determining the direction of induced current or emf.

The direction of any magnetic induction effect is such as to oppose the cause of the effect.

-The "cause" can be changing the flux through a stationary circuit due to varying B, changing flux due to motion of conductors, or both.

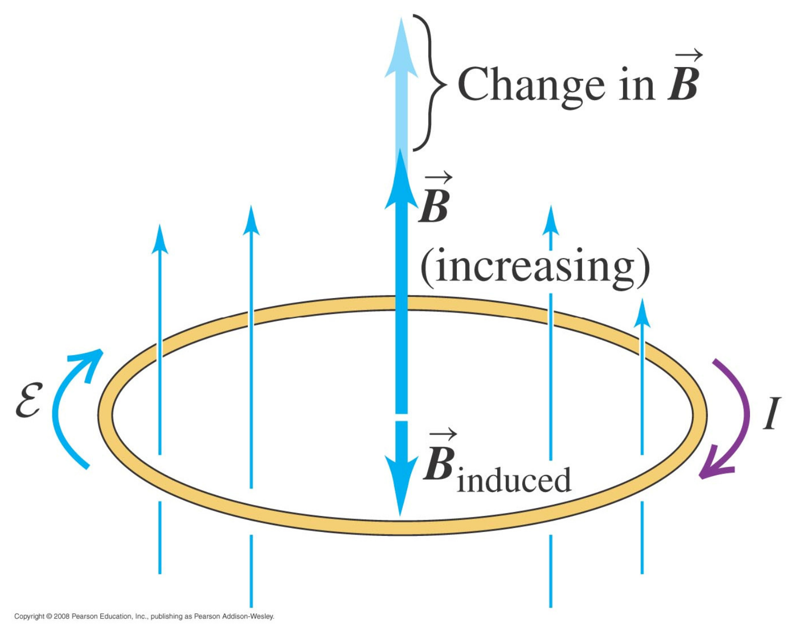


- If the flux in an stationary circuit changes, the induced current sets up a magnetic field *opposite* to the original field if original B *increases*, but in the *same direction* as original B if B *decreases*.

- The induced current opposes the change in the flux through a circuit (not the flux itself).

- If the change in flux is due to the motion of a conductor, the direction of the induced current in the moving conductor is such that the direction of the magnetic force on the conductor is opposite in direction to its motion (e.g. slide-wire generator). The induced current tries to preserve the “status quo” by opposing motion or a change of flux.

B induced downward opposing the change in flux (dΦ/dt). This leads to induced current clockwise.



Lenz's Law and the Response to Flux Changes

- Lenz's Law gives only the *direction* of an induced current. The *magnitude* depends on the circuit's resistance. Large R → small induced I → easier to change flux through circuit.

- If loop is a good conductor → I induced present as long as magnet moves with respect to loop. When relative motion stops → I = 0 quickly (due to circuit's resistance).

- If R = 0 (superconductor) → I induced (*persistent current*) flows even after induced emf has disappeared (after magnet stopped moving relative to loop). The flux through loop is the same as before the magnet started to move → flux through loop of R =0 does not change.

Magnetic levitation:

-The principle of levitation is Lenz' rule.

1) The magnetic field created by the induced current in a metallic sample due to time-fluctuation of the external magnetic field of the coil wants to avoid its cause (i.e., the coil's fluctuating magnetic field).

2) Thus, the induced magnetic field in the sample and the external fluctuating magnetic field of the coil repel each other.

3) The induced magnetic field (and the sample) move away from its cause, i.e. away from the coil's magnetic field. Then, for a conical coil (smaller radius at the bottom than at the top) the metallic sample will move upward due to this levitation force, until the force of gravity balances the force of levitation. (The levitation force is larger at the bottom of the conical coil than at the top of the coil).

Induced Current / Eddy current levitation:

- The rail and the train exert magnetic fields and the train is levitated by repulsive forces between these magnetic fields.
- B in the train is created by electromagnets or permanent magnets, while the repulsive force in the track is created by a induced magnetic field in conductors within the tracks.
- Problems:

(1) at slow speeds the current induced in the coils of the track's conductors and resultant magnetic flux is not large enough to support the weight of the train. Due to this, the train needs wheels (or any landing gear) to support itself until it reaches a speed that can sustain levitation.

(2) this repulsive system creates a field in the track (in front and behind the lift magnets) which act against the magnets and creates a "drag force". This is normally only a problem at low speed.

4. Motional Electromotive Force

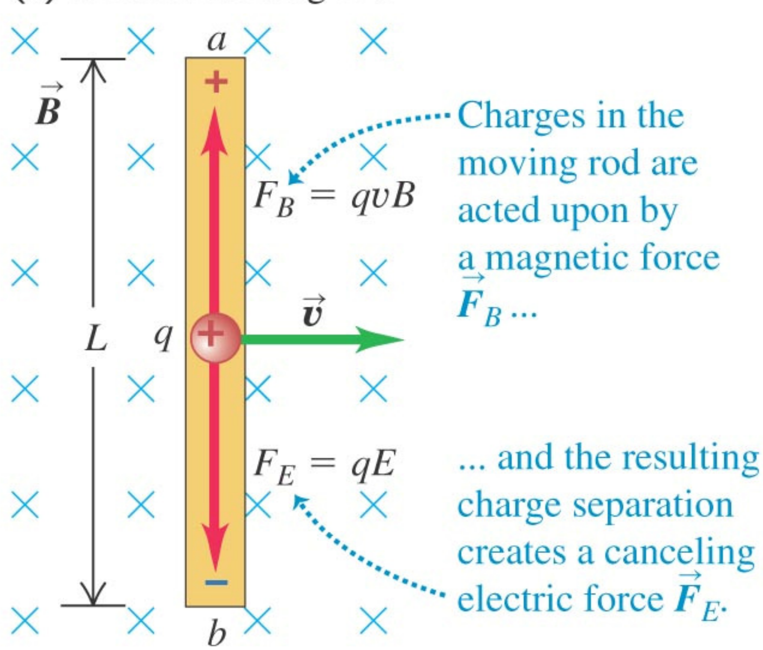
- A charged particle in rod experiences a magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ that causes free charges in rod to move, creating excess charges at opposite ends.

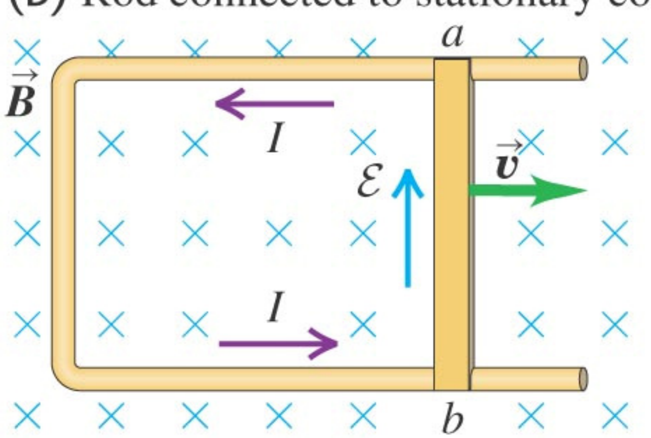
- The excess charges generate an electric field (from a to b) and electric force (F = q E) opposite to magnetic force.

- Charge continues accumulating until F_E compensates F_B and charges are in equilibrium → q E = q v B

$$V_{ab} = E \cdot L = v \cdot B \cdot L$$

- If rod slides along stationary U-shaped conductor → no F_B acts on charges in U-shaped conductor, but excess charge at ends of straight rod redistributes along U-conductor, creating an electric field.





The motional emf $\mathcal{E}$ in the moving rod creates an electric field in the stationary conductor.

-The electric field in stationary U-shaped conductor creates a current → moving rod became a source of emf (motional electromotive force). Within straight rod charges move from lower to higher potential, and in the rest of circuit from higher to lower potential.

$\varepsilon = vBL$ Length of rod and velocity perpendicular to $\vec{B}$.

Induced current: $I = \frac{\varepsilon}{R} = \frac{vBL}{R}$

- The emf associated with the moving rod is equivalent to that of a battery with positive terminal at *a* and negative at *b*.

Motional emf: general form (alternative expression of Faraday's law)

$$d\varepsilon = (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\varepsilon = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$ Closed conducting loop

-This expression can only be used for problems involving <u>moving conductors</u>. When we have stationary conductors in changing magnetic fields, we need to use: $\varepsilon = -d\Phi_B/dt$.

5. Induced Electric Fields

- An induced emf occurs when there is a changing magnetic flux through a stationary conductor.

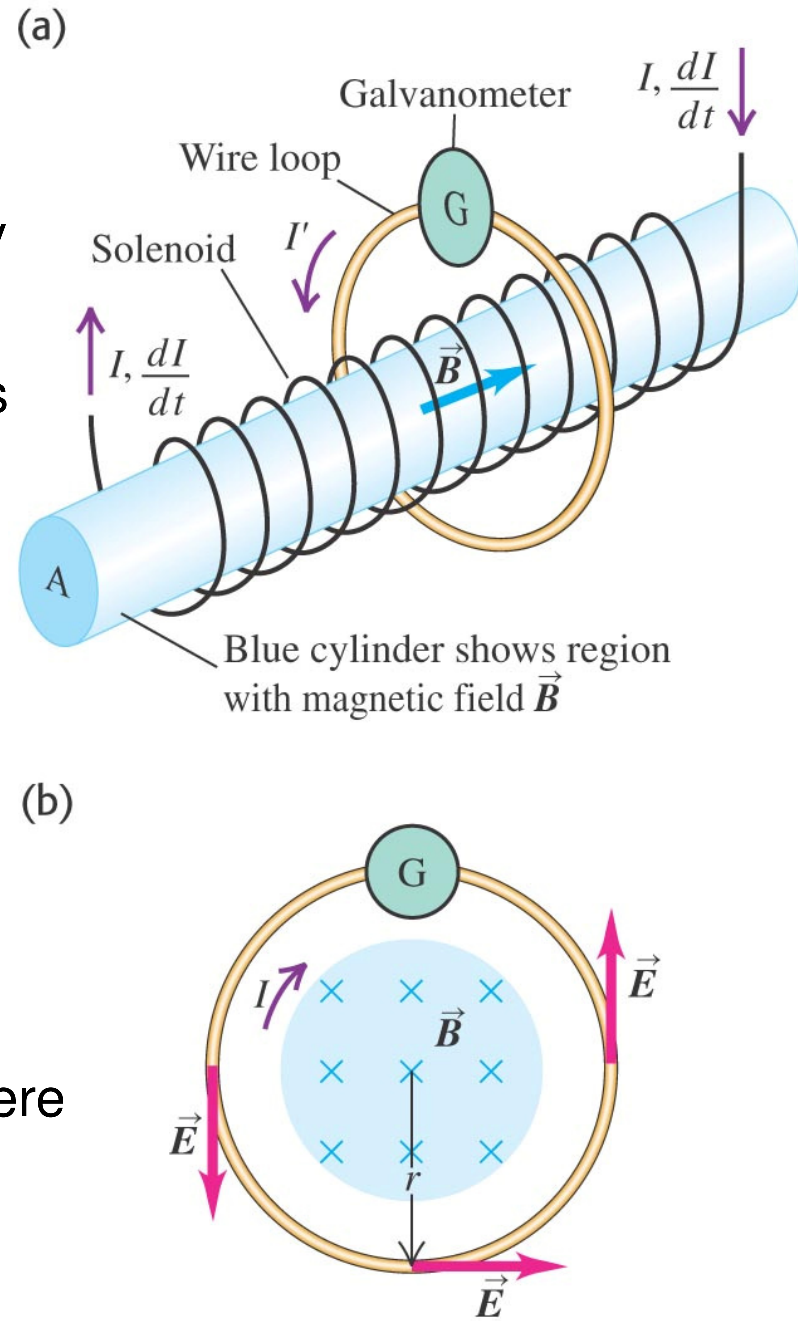
- A current (I) in solenoid sets up B along its axis, the magnetic flux is:

$$\Phi_B = B \cdot A = \mu_0 n I A$$

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\mu_0 n A \frac{dI}{dt}$$

Induced current in loop (I’): I’ = ε / R

- The force that makes the charges move around the loop is not a magnetic force. There is an induced electric field in the conductor caused by a changing magnetic flux.



-The total work done on q by the induced $\vec{E}$ when it goes once around the loop: W = q ε → $\vec{E}$ is not conservative.

Conservative $\vec{E}$ → $\oint \vec{E} \cdot d\vec{l} = 0$

Non-conservative $\vec{E}$ → $\oint \vec{E} \cdot d\vec{l} = \varepsilon = -\frac{d\Phi_B}{dt}$ (stationary integration path)

- Cylindrical symmetry → $\vec{E}$ magnitude constant, direction is tangent to loop.

$$\oint \vec{E} \cdot d\vec{l} = 2\pi \cdot r \cdot E \longrightarrow E = \frac{1}{2\pi r}\left|\frac{d\Phi_B}{dt}\right|$$

- Faraday's law: 1) an emf is induced by magnetic forces on charges when a conductor moves through $\vec{B}$.

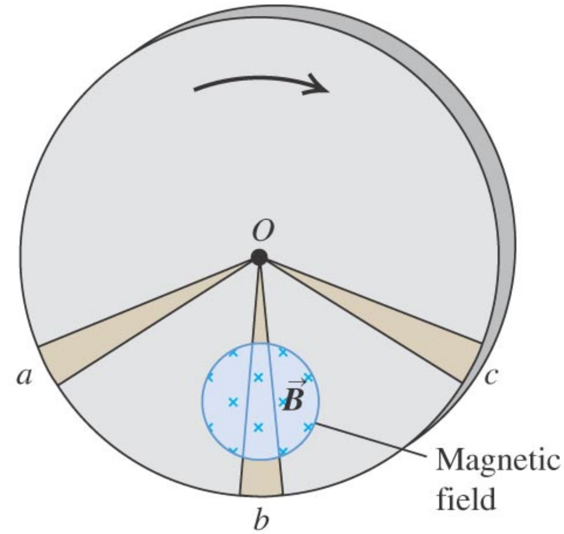
2) a time-varying $\vec{B}$ induces $\vec{E}$ in stationary conductor and emf. $\vec{E}$ is induced even when there is no conductor. Induced $\vec{E}$ is non-conservative, "non-electrostatic". No potential energy associated, but $\vec{F}_E = q\,\vec{E}$.

6. Eddy Currents

- Induced currents that circulate throughout the volume of a material.

Ex.: $\vec{B}$ confined to a small region of rotating disk → Ob moves across $\vec{B}$ and emf is induced → induced circulation of eddy currents.

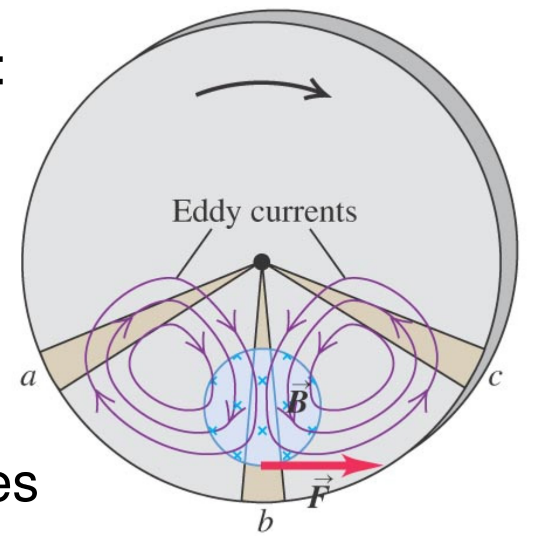
Sectors Oa and Oc are not in B, but provide return conducting paths for charges displaced along Ob to return from b to O.



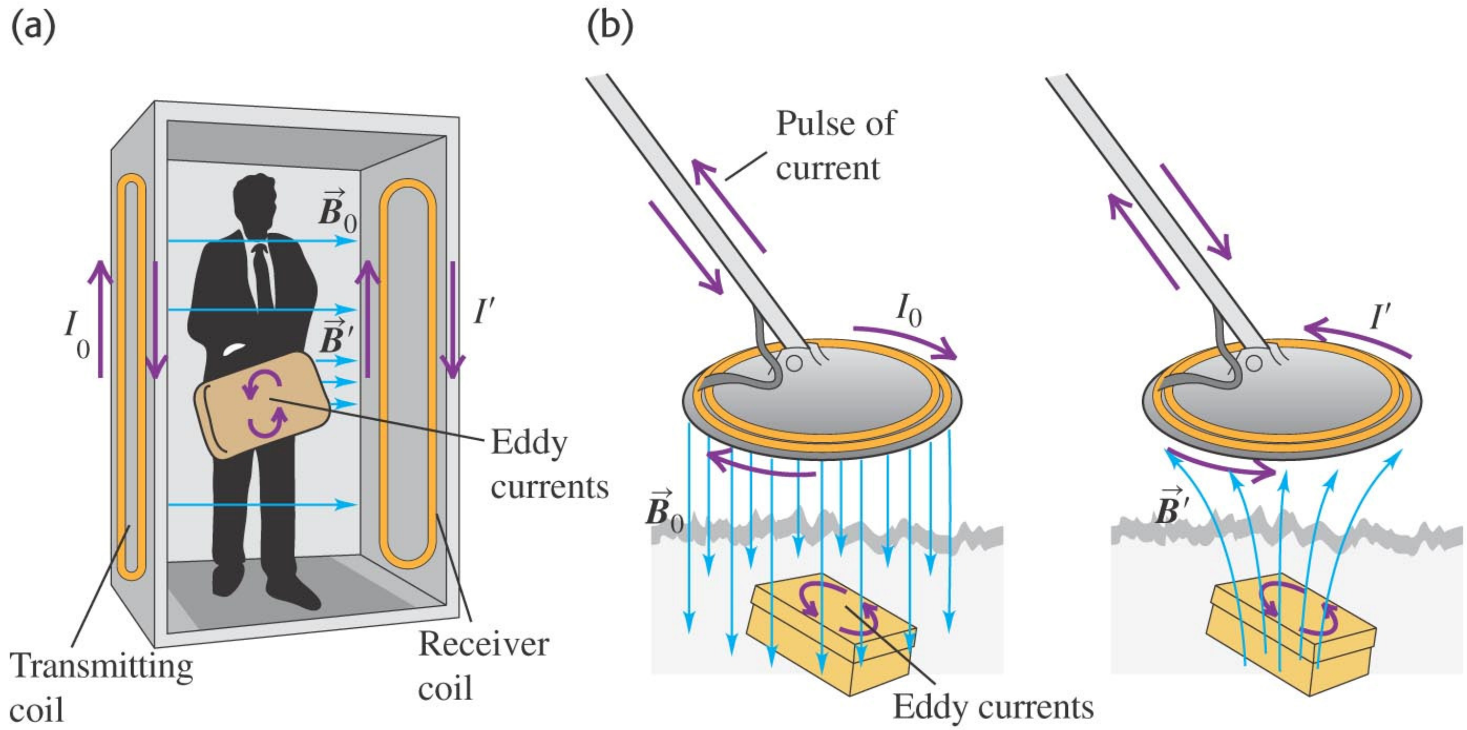
Induced I experiences $\vec{F}_B$ that opposes disk rotation:

$\vec{F} = I\vec{L} \times \vec{B}$ (right) → current and $\vec{L}$ downward.

(the return currents lie outside $\vec{B}$ → do not experience F_B).



-The interaction between eddy currents and B causes braking of disk.



(a) Metal detector (airport security checkpoint) generates an alternating $\vec{B}_0$ that induces eddy currents in conducting object (suitcase). These currents produce alternating $\vec{B}'$ that induces current in detector's receiver (I').

(b) Same principle as (a).

7. Displacement Current and Maxwell's Equations

- A varying electric field gives rise to a magnetic field.

Ampere's Law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$ (incomplete)

Charging a capacitor: conducting wires carry i_C (*conduction current*) into one plate and out of the other, Q and E between plates increase.

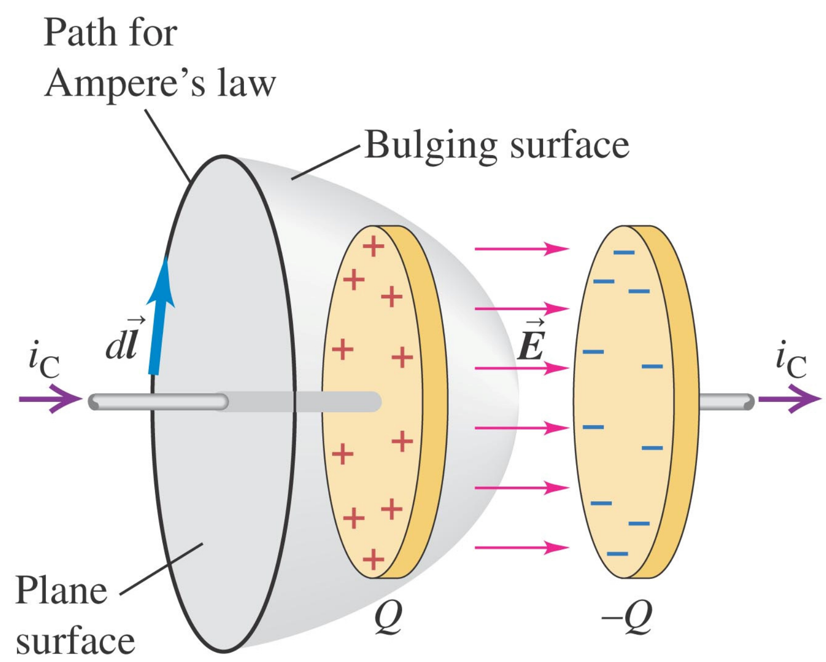
$\oint \vec{B} \cdot d\vec{l} = \mu_0 i_C$ but also = 0 for surface bulging out

Contradiction?

As capacitor charges, E and Φ_E through surface increase.

$$q = C \cdot v = \left(\frac{\varepsilon \cdot A}{d}\right)(E \cdot d) = \varepsilon \cdot E \cdot A = \varepsilon \cdot \Phi_E$$

$$i_C = \frac{dq}{dt} = \varepsilon \frac{d\Phi_E}{dt}$$

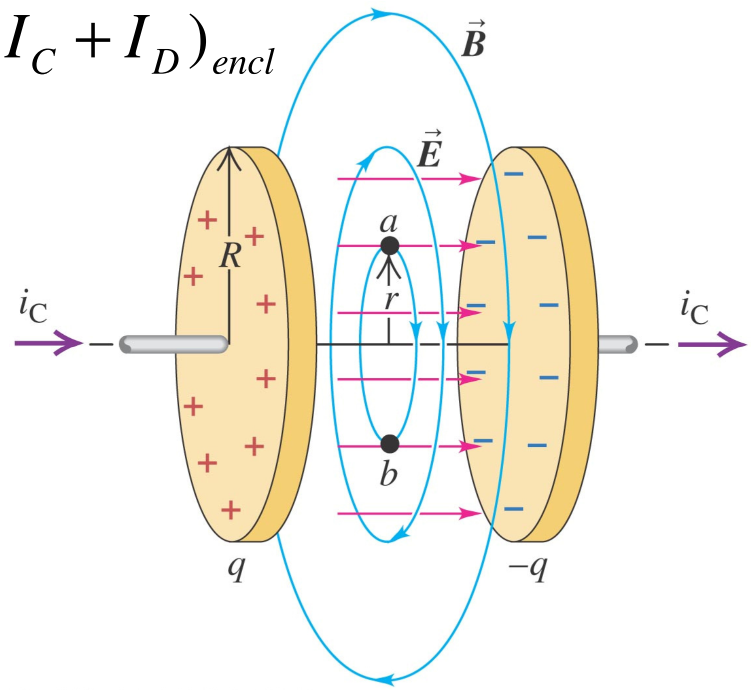


Displacement current (i_D): fictitious current in region between capacitor's plates.

$$i_D = \varepsilon \frac{d\Phi_E}{dt}$$

Changing the flux through curved surface is equivalent in Ampere's law to a conduction current through that surface (i_D)

Generalized Ampere's Law: $$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_C + I_D)_{encl}$$



Valid for any surface we use: for curved surface $i_c = 0$, for flat surface $i_D = 0$.
i_c (flat surface) = i_D (curved surface)

Displacement current density (j_D):

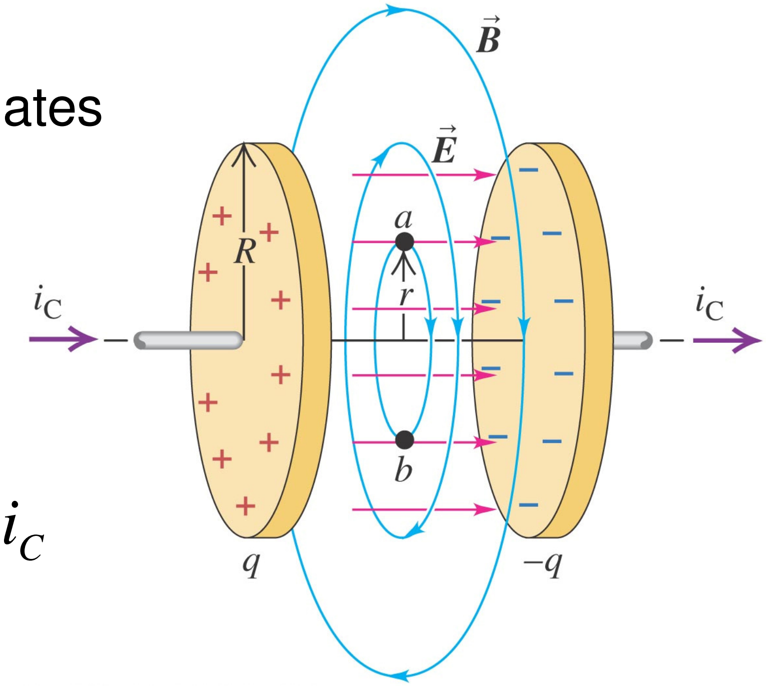
$$j_D = \frac{i_D}{A} = \varepsilon \frac{dE}{dt}$$

The displacement current is the source of B in between capacitor's plates. It helps us to satisfy Kirchoff's junction's rule: I_C in and I_D out

The reality of Displacement Current

- Displacement current creates B between plates of capacitor while it charges.

(r < R)



$$\oint \vec{B}\cdot d\vec{l} = 2\pi \cdot r \cdot B = \mu_0 I_{encl} = \mu_0 j_D A$$

$$= \mu_0\left(\frac{i_D}{\pi \cdot R^2}\right)(\pi \cdot r^2) = \mu_0 \frac{r^2}{R^2} i_D = \mu_0 \frac{r^2}{R^2} i_C$$

$$B = \frac{\mu_0}{2\pi}\frac{r}{R^2} i_C$$

- In between the plates of the capacitor: (r < R) → B = 0 at r = 0 (axis) and increases linearly with distance from axis.

- For r> R → B is same as though the wire were continuous and plates not present.

Maxwell's Equations of Electromagnetism

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Gauss Law for $\overrightarrow{E}$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Gauss Law for $\overrightarrow{B}$ **(there are no magnetic monopoles)**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i_C + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl}$$

Ampere's law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

Faraday's law

$$\vec{E} = \vec{E}_c + \vec{E}_n$$

Total electric field = E caused by a distribution of charges at rest (E_c = electrostatic) + E magnetically induced (E_n, non-electrostatic).

Symmetry in Maxwell's Equations

In empty space $i_c = 0$, $Q_{encl} = 0$

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{l} = \varepsilon_o \mu_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$$

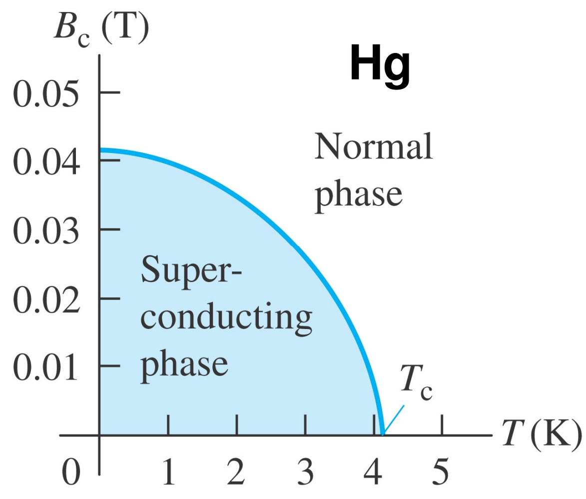
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

8. Superconductivity

- Sudden disappearance of R when material cooled below critical T (T_c).

- For superconducting materials T_c changes when material under external B_0.



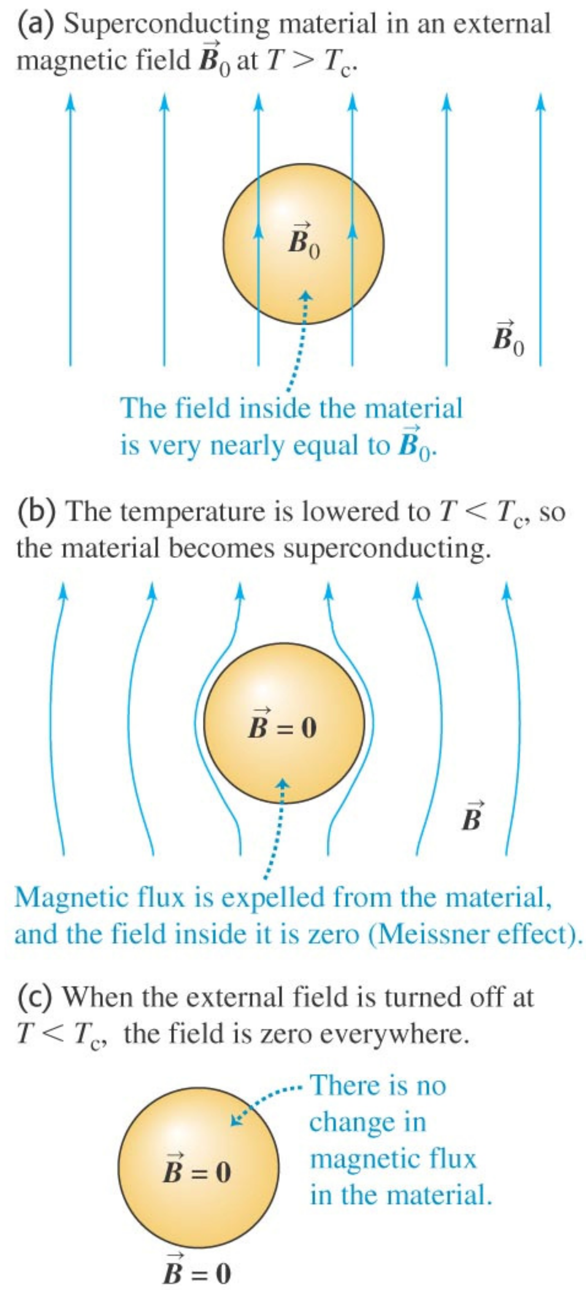
- With increasing B_0, superconducting transition occurs at lower and lower T.

- Critical field (B_c): minimum B required to eliminate SC at T<Tc.

The Meissner effect

- SC sphere in B_0 at T > T_c → normal phase (not SC)

- If T < T_c and B_0 not large enough to prevent SC transition → distortion of field lines, no lines inside material.



- If coil wrapped around sphere → emf induced in coil shows that during SC transition the magnetic flux through coil decreases (to zero).

- If B_0 turned off while material still in SC phase → no emf induced in coil and no B outside sphere.

- During SC transition in the presence of B_0, all magnetic flux is expelled from the bulk of the sphere and the magnetic flux through a coil is zero. The “expulsion” of B is the “Meissner effect”. That results in an increased B (more densely packed field lines) close to the sides of the sphere.

http://www.youtube.com/watch?v=G3eI4SVDyME

(11:11) Emf Induction while moving a bar magnet over a conducting loop

http://www.youtube.com/watch?v=qxuGDEz8wDg&NR=1

(10:35), (17:10)
Emf induction while changing the angle phi in a loop

(38:28) Eddy currents

ALTERNATING CURRENT

Alternating current

As we have seen earlier a rotating coil in a magnetic field, induces an alternating emf and hence an alternating current. Since the emf induced in the coil varies in magnitude and direction periodically, it is called an alternating emf. The significance of an alternating emf is that it can be changed to lower or higher voltages conveniently and efficiently using a transformer. Also the frequency of the induced emf can be altered by changing the speed of the coil. This enables us to utilize the whole range of electromagnetic spectrum for one purpose or the other. For example domestic power in India is supplied at a frequency of 50 Hz. For transmission of audio and video signals, the required frequency range of radio waves is between 100 KHz and 100 MHz. Thus owing to its wide applicability most of the countries in the world use alternating current.

Measurement of AC

Since alternating current varies continuously with time, its average value over one complete cycle is zero. Hence its effect is measured by rms value of a.c.

RMS value of a.c.

The rms value of alternating current is defined as that value of the steady current, which when passed through a resistor for a given time, will generate the same amount of heat as generated by an alternating current when passed through the same resistor for the same time.

The rms value is also called effective value of an a.c. and is denoted by I_{rms} when an alternating current i=Io sin ωt flows through a resistor of resistance R, the amount of heat produced in the resistor in a small time dt is dH = i^2 R dt. We know that alternating current is given I = $i_0$sinωt

The total amount of heat produced in the resistance in one complete cycle is

$$H = \int_0^T i^2\, Rdt$$

$$H = \int_0^T i_0^2 sin^2\omega t\, Rdt$$

$$H = i_0^2 R \int_0^T \left(\frac{1 - cos2\omega t}{2}\right) dt$$

$$H = \frac{i_0^2 R}{2}\left[\int_0^T dt - \int_0^T cos2\omega tdt\right]$$

As $\int_0^T cos2\omega tdt = 0$

$$H = \frac{i_0^2 R}{2} T$$

But this heat is also equal to the heat produced by rms value of AC in the same resistor (R) and in the same time (T),

$$H = I_{rms}^2 RT$$

Thus

$$I_{rms}^2 RT = \frac{i_0^2 R}{2} T$$

$$\therefore I_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

We can calculate rms value as root mean square value :
The mean value or average value of ac over time T is given by

$$i_{rms}^2 = \frac{\int_0^T i^2 dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{\int_0^T i_0^2 sin^2(\omega t) dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{i_0^2 \int_0^T [1 - cos 2\omega t] dt}{2T}$$

As $\int_0^T cos 2\omega t dt = 0$

$$i_{rms}^2 = \frac{i_0^2 T}{2T} = \frac{i_0^2}{2}$$

$$\therefore i_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

Similarly

$$E_{rms} = \frac{E_0}{\sqrt{2}}$$

Solved numerical

Q) If the voltage in ac circuit is represented by the equation

$$V = 220\sqrt{2}sin(314t - \varphi)$$

Calculate (a) peak and rms value of the voltage
(b) average voltage
(c) frequency of ac

<u>Solution:</u>

(a) For ac voltage
$V = V_0 \sin(\omega t - \varphi)$
The peak value
$V_0 = 220\sqrt{2} = 311V$
The rms value of voltage

$$V_{rms} = \frac{V_0}{\sqrt{2}} = \frac{220\sqrt{2}}{\sqrt{2}} = 220V$$

(b) Average voltage in full cycle is zero, Average voltage in half cycle is

$$V_{ave} = \frac{2}{\pi}V_0 = \frac{2}{\pi}311 = 198.71V$$

(c) As $\omega = 2\pi f$
$2\pi f = 314$
$f = 314/2\pi = 50\text{Hz}$

Q) Write the equation of a 25 cycle current sine wave having rms value of 30 A.

<u>Solution:</u>

Given: frequency f = 25 HZ and I_{rms} = 30A or $i_0 = 30\sqrt{2}$
$I = i_0 \sin(2\pi f)t$
$I = 30\sqrt{2}\sin(2\pi \times 25)t$
$I = 30\sqrt{2}\sin(50\pi)t$

Q) An electric current has both A.C. and D.C. components. The value of the D.C component is equal to 12A while the A.C. component is given as I = 9sinωt A. Determine the formula for the resultant current and also calculate the value of I_{rms}

Solution: Resultant current at any instant of time will be I = 12+9sinωt

Now $I_{rms} = \sqrt{\langle I^2 \rangle} = \sqrt{(12 + 9\sin\omega t)^2}$

$I_{rms} = \sqrt{\langle 144 + 216 sin\omega t + 81 sin^2\omega t \rangle}$

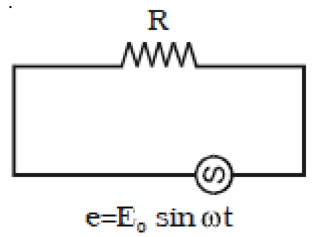
Here, the average is taken over a time interval equal to the periodic time
Now <144> = 144
216<sinωt> = 0
And 81<$\sin^2$ ωt> = 81 × (1/2) = 40.5

$\therefore I_{rms} = \sqrt{144 + 40.5} = 13.58\ A$

Series AC Circuit

1) When only resistance is in an ac circuit

Consider a simple ac circuit consisting of resistor of resistance R and and ac generator, as shown in figure



According to Kirchhoff's loop law at any instant , the algebraic sum of the potential difference around a closed loop in a circuit must be zero

$e - V_R = 0$

$e - I_R R = 0$

$E_0 \sin\omega t - I_R R = 0$

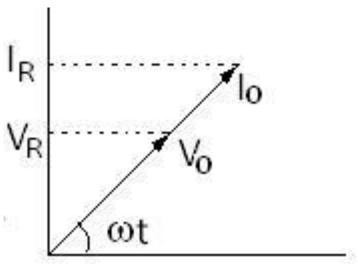
$I_R = E_0 \sin\omega t / R = I_0 \sin\omega t$ ---(1)

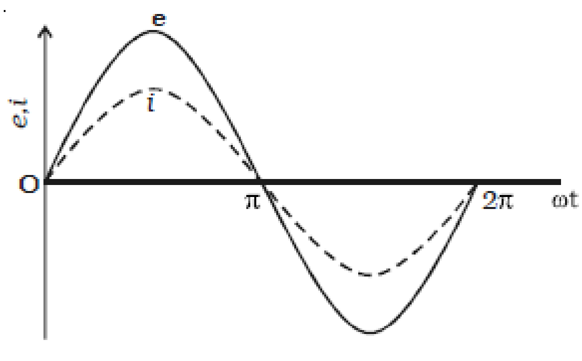
Where I_0 is the maximum current $I_0 = E_0/R$

From above equations, we see that the instantaneous voltage drop across the resistor is

$V_R = I_O R \sin\omega t$ ---(2)

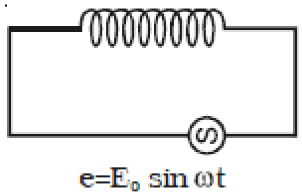
We see in equation (1) and (2) I_R and V_R both vary as sinωt and reach their maximum values at the same time as shown in graph they are said to be in phase.





2) When only Inductor is in an ac circuit

Consider an ac circuit consisting only of an indiuctor of inductance l connected to the termicals of ac generator, as shown in figure



The induced emf across the inductor is given by L(di/dt). On applying Kirchhoff's loop rule to the circuit

$$e - V_L = 0$$
$$e = L\frac{di}{dt}$$
$$E_0 sin\omega t = L\frac{di}{dt}$$

Integrating above expression as a function of time

$$i_L = \frac{E_0}{L}\int sin\omega t dt = -\frac{E_0}{\omega L}cos\omega t + C$$

For average value of current over one time period to be zero, C= 0

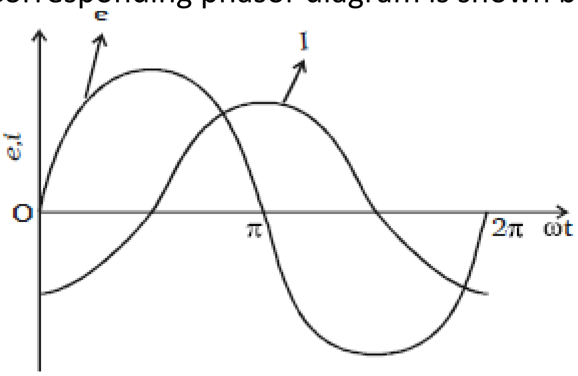
$$\therefore i_L = -\frac{E_0}{\omega L}cos\omega t$$

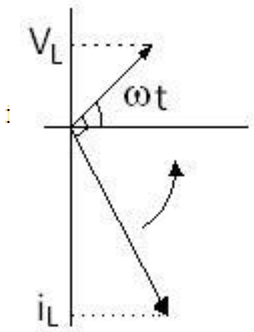
When we use the trigonometric identity $cos\omega t \;=\; -sin\left(\omega t - \frac{\pi}{2}\right)$

We can express equation as

$$i_L = \frac{E_0}{\omega L}sin\left(\omega t - \frac{\pi}{2}\right)$$

From above equation it is clear that current lags by π/2 to voltage. The voltage reaches maximum , one quarter of than oscillation period before current reaches maximum value. Corresponding phasor diagram is shown below





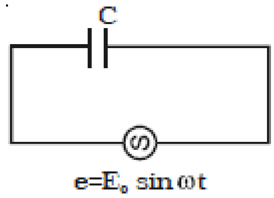
Secondly current is maximum when cosωt = 1

$$i_0 = \frac{E_0}{\omega L}$$

ωL is known as inductive reactance denoted by X_L

3) When only capacitor is in an ac circuit

Figure shows an ac circuit consisting of a capacitor of capacitance C connected across the terminals of an ac generator.



On applying Kirchhoff's rule to this circuit, we get

$$e - V_C = 0$$
$$V_C = e$$
$$V_C = E_0 sin\omega t$$

Where V_C is the instantaneous voltage drop across the capacitor. From the definition of capacitance $V_C = Q/C$, and this value of V_C substituted into equation gives

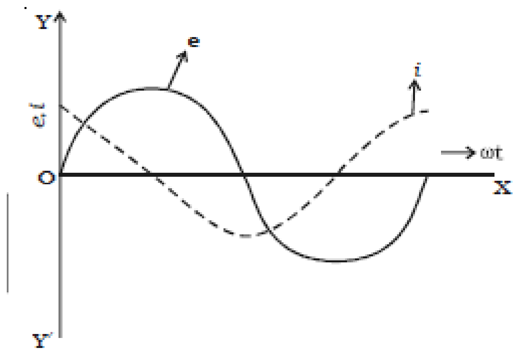
$Q = C E_0 sin\omega t$

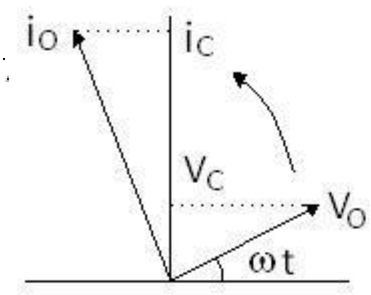
Since i = dQ/dt, on differentiating above equation gives the instantaneous current in the circuit

$$i_C = \frac{dQ}{dt} = CE_0\omega\cos\omega t$$

From above equation it is clear that current leads the voltage by π/2

A plot of current and voltage versus times , shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches maximum value. The corresponding phasor diagram is shown





Secondly when cosωt = 1, in equation $i_C = CE_0\omega\cos\omega t$ the current in circuit is maximum

$$i_C = CE_0\omega = \frac{E_0}{X_C}$$

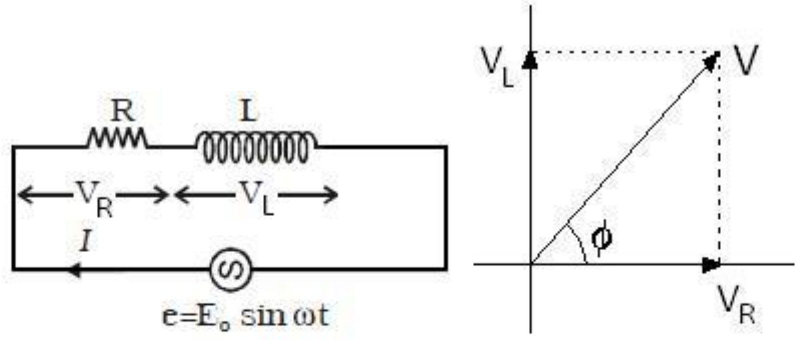
X_C is called the capacitive reactance

$$X_C = \frac{1}{\omega C}$$

For DC supply, ω = 0 therefore X_C will be infinite and current will not flow through capacitor once it is fully charged.

SERIES L-R Circuit

Now consider an ac circuit consisting of a resistor or resistance R and an inductor of inductance L in series with an ac source generator



Suppose in phasor diagram, current is taken along positive direction. The V_R is also along positive x-direction as there is no phase difference between i_R and V_R . While V_L will be along y direction as we know that current lags behind the voltage by 90^O

So we can write

$V = V_R + jV_L$

$V = i_R R + j(iX_L)$

$V= iZ$

Here $Z = R + jX_L = R + j(\omega L)$ is called as impedance of the circuit. Impedance plays the same role in ac circuit as the ohmic resistance does in DC circuit. The modulus of impedance is

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

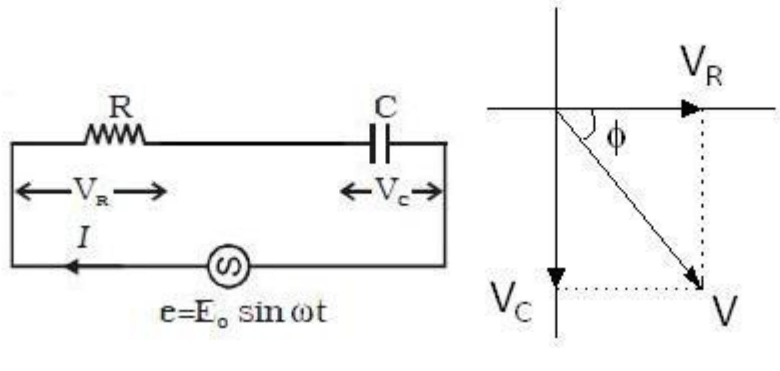
The potential difference leads the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_L}{V_R}\right| = tan^{-1}\left(\frac{X_L}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{\omega L}{R}\right)$$

SERIES R-C Circuit

Now consider an ac circuit consisting of resistance R and a capacitor of capacitance C in series with an ac source generator



Suppose in phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction but V_C is along negative y-direction as current leads the potential by phase 90^O so we can write

$V = V_R - jV_C$

$$V = iR - j\left(\frac{i}{\omega C}\right) = iZ$$

Here impedance

$$Z = R - j\left(\frac{1}{\omega C}\right)$$

And the potential difference lags the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_C}{V_R}\right| = tan^{-1}\left(\frac{X_C}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{1/\omega C}{R}\right) = tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Solved Numerical

Q) An alternating current voltage of 220 V r.m.s. at frequency of 40 cycles/ second is supplied to a circuit containing a pure inductance of 0.01H and a pure resistance of 6 ohm in series. Calculate (i) the current (ii) potential difference across the resistance (iii)potential difference across the inductance (iv) the time lag

Solution

The impedance of L-R circuit is given by

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

$$|Z| = \sqrt{R^2 + (2\pi f L)^2}$$

$$|Z| = \sqrt{(6)^2 + (2 \times 3.14 \times 40 \times 0.01)^2}$$

Z=6.504 ohms

(i) r.m.s value of current

$$i_{rms} = \frac{E_{rms}}{Z} = \frac{220}{6.504} = 33.83\ amp$$

(ii) The potential difference across the resistance is given by

$V_R = i_{rms} \times R = 33.83 \times 6 = 202.98$ Volt

(iii) Potential difference across inductance is given by

$V_L = i_{rms} \times (\omega L) = 33.83 \times 6 = 202.98$ volts

(iv) Phase angle

$$\varphi = tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{2\pi f L}{R}\right) = tan^{-1}\left(\frac{2 \times 3.14 \times 40 \times 0.01}{R}\right)$$

$$\varphi = tan^{-1}(0.4189) = 22^o 73'$$

Now time lag =

$$\frac{\varphi}{360}\times T=\frac{\varphi}{360}\times\frac{1}{f}=\frac{22^{o}73'}{360\times 40}=0.001579\ s$$

Q) An ac source of an angular frequency ω is fed across a resistor R and a capacitor C in series. The current registered is i. If now the frequency of source is changed to ω/3 (but maintaining the same voltage), the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency

Solution:

At angular frequency ω , the current in R-C circuit is given by

$$i_{rms}=\frac{E_{rms}}{\sqrt{R^2+\left(\frac{1}{\omega^2C^2}\right)}}\ ---eq(1)$$

When frequency changed to ω/3 , the current is halved. Thus

$$\frac{i_{rms}}{2}=\frac{E_{rms}}{\sqrt{\left\{R^2+\left(\frac{1}{\frac{\omega^2C^2}{3^2}}\right)\right\}}}$$

$$\frac{i_{rms}}{2}=\frac{E_{rms}}{\left\{R^2+\frac{9}{\omega^2C^2}\right\}}---eq(2)$$

From above equation (1) and (2) we have

$$\frac{E_{rms}}{\sqrt{R^2+\left(\frac{1}{\omega^2C^2}\right)}}=\frac{2E_{rms}}{\sqrt{\left\{R^2+\left(\frac{9}{\omega^2C^2}\right)\right\}}}$$

Solving the equation we get

$$3R^2=\frac{5}{\omega^2C^2}$$

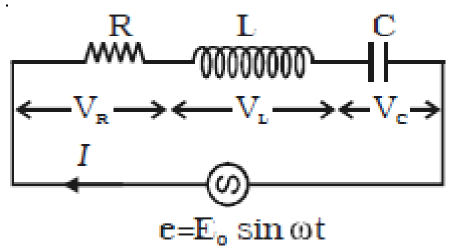
Hence ratio of reactance to resistance

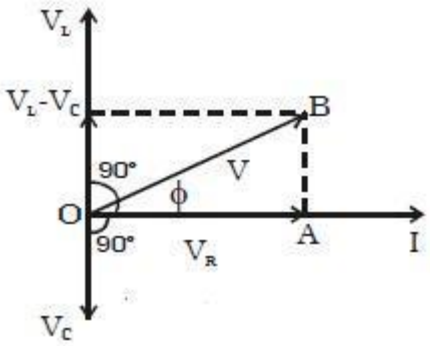
$$\frac{1/_{\omega C}}{R}=\sqrt{\frac{3}{5}}$$

SERIES L – C – R CIRCUIT

Consider an ac circuit consisting of resistance R, capacitor of capacitance C and an inductor of inductance L are in series with ac source generator

Suppose in a phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction, V_L along positive y-direction and V_C along negative y-direction, as potential difference across an inductor leads the current by 90^O in phase while that across a capacitor, lags by 90^O





$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

We can write V = V_R + jV_L – jV_C

V = iR +j(iX_L) – j(iX_C)

V = iR +j[i(X_L – X_C)] = iZ

Here impedance is

$$Z = R + j(X_L - X_C)$$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The potential difference leads the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_L - V_C}{V_R}\right|$$

$$\varphi = tan^{-1}\left|\frac{X_L - X_C}{R}\right|$$

$$\varphi = tan^{-1}\left|\frac{\omega L - \frac{1}{\omega C}}{R}\right|$$

The steady current is given by

$$i = \frac{V_O}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} sin(\omega t + \varphi)$$

The peak current is

$$i_0 = \frac{V_O}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

It depends on angular frequency ω of ac source and it will be maximum when

$$\omega L - \frac{1}{\omega C} = 0$$

$$\omega = \sqrt{\frac{1}{LC}}$$

And corresponding frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This frequency is known as resonant frequency of the given circuit. At this frequency peak current will be $i_0 = \frac{V_0}{R}$

This resistance R in the LCR circuit is zero, the peak current at resonance is $i_0 = \frac{V_0}{R}$

It means, there can be a finite current in pure LC circuit even without any applied emf. When a charged capacitor is connected to pure inductor

This current in the circuit is at frequency $f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

Solved Numerical

Q) A resistor of resistance R, an inductor of inductance L and a capacitor of capacitance C all are connected in series with an a.c. supply. The resistance of R is 16 ohm. And for a given frequency, the inductive reactance of L is 24 ohms and capacitive reactance of C is 12 ohms. If the current in circuit is 5amp, find

(a) The potential difference across R, L and C

(b) the impedance of the circuit

(c) the voltage of ac supply

(d) Phase angle

Solution:

(a) Potential difference across resistance V_R = iR = 5×16 =80 volt

Potential difference across inductance

V_L = i × (ωL) = 5 × 24 = 120 volt

Potential across condenser

V_C = i × (1/ωC) = 5 × 12 = 60 volts

(b) Impedance

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$|Z| = \sqrt{16^2 + (24-12)^2} = 20\ ohm$$

(c) The voltahe of ac supply is given by

V = iZ = 5 ×20 = 100 volt

(c) Phase angle

$$\varphi = tan^{-1}\left|\frac{\omega L - \frac{1}{\omega C}}{R}\right|$$

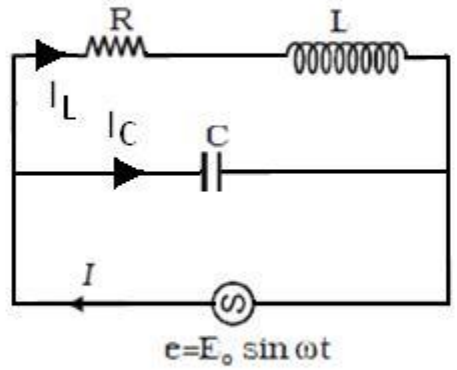
$$\varphi = tan^{-1}\left|\frac{24 - 12}{16}\right|$$

$$\varphi = tan^{-1}(0.75) = 36^{o}87'$$

PARALLEL AC CIRCUIT

Consider an alternating source connected across an inductor L in parallel with a capacitance C

The resistance in series with the inductance is R and with the capacitor as zero



Let the instantaneous value of emf applied be V and the corresponding current is I, I_L and I_C. Then

$I = I_L + I_C$

Or

$$\frac{V}{Z} = \frac{V}{R + j\omega L} - \frac{V}{\frac{j}{\omega C}}$$

$$\frac{V}{Z} = \frac{V}{R + j\omega L} + j\omega CV \ \ (as\ j^2 = -1)$$

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + j\omega C$$

$\frac{1}{Z}$ is called admittance Y

$$\frac{1}{Z} = Y = \frac{1}{R + j\omega L}\frac{R - j\omega L}{R - j\omega L} + j\omega C$$

$$Y = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$Y = \frac{R + j(\omega C R^2 + \omega^3 L^2 C - \omega L)}{R^2 + \omega^2 L^2}$$

Magnitude of admittance

$$|Y| = \frac{\sqrt{R^2 + (\omega C R^2 + \omega^3 L^2 C - \omega L)^2}}{R^2 + \omega^2 L^2}$$

The admittance will be minimum. When
$\omega CR^2 + \omega^3 L^2 C - \omega L = 0$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

It gives the condition of resonance and corresponding frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is known as resonance frequency. At resonance frequency admittance is minimum or impedance is maximum. Thus the parallel circuit does not allow this frequency from source to pass in the circuit. Due to this reason the circuit with such frequency is known as rejecter circuit

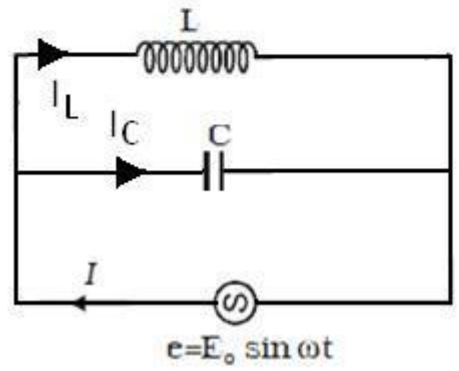
Note

If R = 0, resonance frequency

$f = \frac{1}{2\pi\sqrt{LC}}$ is same as resonance frequency in series circuit

Solved numerical

Q) For the circuit shown in figure. Current is inductance is 0.8A while in capacitance is 0.6A. What is the current drawn from the source



<u>Solution:</u>

In this circuit $E = E_0 \sin\omega t$ is applied across an inductance and capacitance in parallel, current in inductance will lag the applied voltage while across the capacitor will lead and so

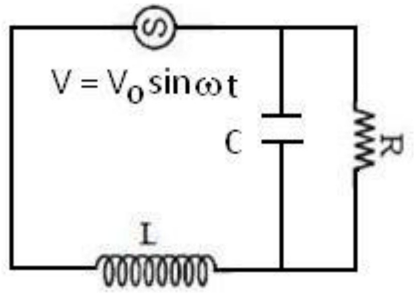
$$i_L = \frac{V}{X_L}\sin\left(\omega t - \frac{\pi}{2}\right) = -0.8\cos\omega t$$
$$i_C = \frac{V}{X_C}\sin\left(\omega t + \frac{\pi}{2}\right) = +0.6\cos\omega t$$

So current from the source

$i = i_L + i_C = -0.2\cos\omega t$

$|i_0| = 0.2A$

Q) An emf $V_0 \sin\omega t$ is applied to a circuit which consists of self-inductance L of negligible resistance in series with a variable capacitor C. the capacitor is shunted by a variable resistance R. Find the value of C for which the amplitude of the current is independent of R
Solution



Solution:
First we will calculate impedance of the circuit
The complex impedance of the circuit
Z = jωL +Z'
Here Z' is complex impedance of parallel combination of Capacitor and R

$$\frac{1}{Z'} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

$$Z' = \frac{R}{1 + j\omega CR} = \frac{R(1 - j\omega CR)}{(1 + j\omega CR)(1 - j\omega CR)} = \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$Z = \mathrm{j\omega L} + \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$Z = \mathrm{j\omega L} + \frac{R}{1 + \omega^2 C^2 R^2} - \frac{j\omega CR^2}{1 + \omega^2 C^2 R^2}$$

$$Z = j\left(\mathrm{\omega L} - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right) + \frac{R}{1 + \omega^2 C^2 R^2}$$

Magnitude of Z is given by

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 + \left(\mathrm{\omega L} - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right)^2$$

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} + \left(\frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right)^2$$

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 (1 + \omega^2\ C^2\ R^2) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2}$$

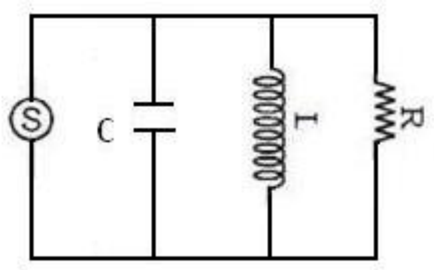
$$Z^2 = \left(\frac{R^2}{1 + \omega^2 C^2 R^2}\right) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2}$$

The value of current will be independent of R . It is possible when

$$R^2 - 2\omega^2 LCR^2 = 0$$

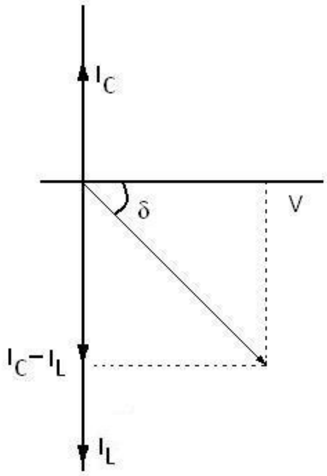
$$C = \frac{1}{2}\omega^2 L$$

Q) Derive the expression for the total current flowing in the circuit using phaser diagram



Solution:

The phasor diagram of the voltage and current is as shown in figure. In order to obtain the total current, we shall have to consider the addition of the currents. From the diagram we have



$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

But

$$I_R = \frac{V}{R} \,; I_L = \frac{V}{X_L} ; I_C = \frac{V}{X_C}$$

$$I = V\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_L} - \frac{1}{X_C}\right)^2}$$

From figure

$$tan\delta = \frac{I_L - I_C}{I_R} = \frac{\frac{1}{X_L} - \frac{1}{X_C}}{\frac{1}{R}}$$

$$tan\delta = R\left(\frac{1}{X_L} - \frac{1}{X_C}\right)$$

Q-factor

The selectivity or sharpness of a resonant circuit is measured by the quality factor or Q factor. In other words it refers to the sharpness of tuning at resonance. The Q factor of a series resonant circuit is defined as the ratio of the voltage across a coil or capacitor to the applied voltage.

$$Q = \frac{voltage\ across\ L\ or\ C}{applied\ voltage} --(1)$$

Voltage across L = I ω_oL ...(2)

where ω_o is the angular frequency of the a.c. at resonance. The applied voltage at resonance is the potential drop across R, because the potential drop across L is equal to the drop across C and they are 180° out of phase. Therefore they cancel out and only potential drop across R will exist.

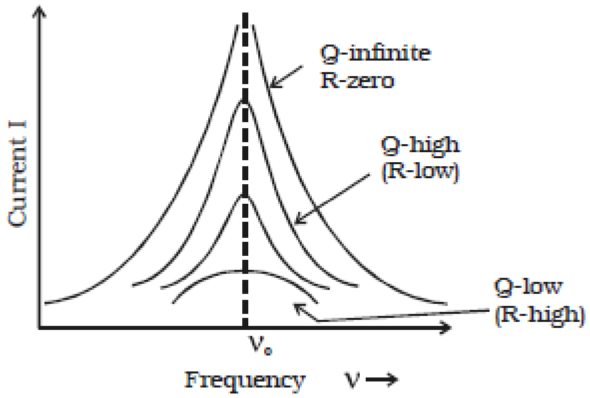
Applied Voltage = IR ...(3)

Substituting equations (2) and (3) in equation (1)

$$Q = \frac{\mathrm{I\ \omega oL}}{\mathrm{IR}} = \frac{\mathrm{\omega oL}}{\mathrm{R}}$$

$$Q = \frac{1}{\sqrt{RC}}\frac{L}{R}$$

Q is just a number having values between 10 to 100 for normal frequencies. Circuit with high Q values would respond to a very narrow frequency range and vice versa. Thus a circuit with a high Q value is sharply tuned while one with a low Q has a flat resonance. Q-factor can be increased by having a coil of large inductance but of small ohmic resistance. Current frequency curve is quite flat for large values of resistance and becomes more sharp as the value of resistance decreases. The curve shown in graph is also called the frequency response curve.



Sharpness of resonance

The amplitude of the current in the series *LCR* circuit is given by

$$i_{max} = \frac{v_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and is maximum when $\omega = \omega_0 = 1/\sqrt{(LC)}$ The maximum value is $i_{max} = V_{max}/R$

For values of ω other than ω0, the amplitude of the current is less than the maximum value.

Suppose we choose a value of ω for which the current amplitude is $1/\sqrt{2}$ times its maximum value. At this value, the power dissipated by the circuit becomes half. From the curve in Fig. ,we see that there are two such values of ω, say, ω_1 and ω_2, one greater and the other smaller than ω_0 and symmetrical about ω_0. We may write

$\omega_1 = \omega_0 + \Delta\omega$

$\omega_2 = \omega_0 - \Delta\omega$

The difference $\omega_1 - \omega_2 = 2\Delta\omega$ is often called the *bandwidth* of the circuit. The quantity $(\omega_0 / 2\Delta\omega)$ is regarded as a measure of the sharpness of resonance. The smaller the $\Delta\omega$, the sharper or narrower is the resonance.

We see from Fig. that if the resonance is less sharp, not only is the maximum current less, the circuit is close to resonance for a larger range $\Delta\omega$ of frequencies and the tuning of the circuit will not be good. So, less sharp the resonance, less is the selectivity of the circuit or vice versa.

Value of $\Delta\omega = \frac{R}{2L}$

we see that if quality factor is large, i.e., R is low or L is large, the circuit is more selective.

POWER IN AN AC CIRCUIT

In case of steady current the rate of doing work is given by,

P = VI

In an alternatin circuit, current and voltage both vary with time, so the work done by the source in time intrerval dt is given by

dw = Vidt

Suppose in an ac, the current is leading the voltage by an angle φ . Then we can write

$V = V_m \sin\omega t$ and

$I = i_m \sin(\omega t+\varphi)$

$dw = V_m i_m \sin\omega t \sin(\omega t+\varphi)dt$

$dw = V_m i_m (\sin^2 \omega t\cos\varphi + \sin\omega t\cos\omega t\sin\varphi)\, dt$

The total work done in a complete cycle is

$$W = V_m i_m \cos\varphi \int_0^T \sin^2 \omega t dt + V_m i_m \sin\varphi \int_0^T \sin\omega t\cos\omega t dt$$

$$W = \frac{1}{2} V_m i_m \cos\varphi \int_0^T (1 - \cos 2\omega t)\, dt + \frac{1}{2} V_m i_m \sin\varphi \int_0^T \sin 2\omega t dt$$

$$W = \frac{1}{2} V_m i_m T \cos\varphi$$

The average power delivered by the source is, therefore

P=W/T

$$P = \frac{1}{2} V_m i_m \cos\varphi$$

$$P = \frac{V_m}{\sqrt{2}} \frac{i_m}{\sqrt{2}} \cos\varphi$$

$$P = V_{rms} i_{rms} \cos\varphi$$

This can also be written as,

$P = I^2 Z\cos\phi$

Here, Z is impedance, the term $\cos\varphi$ is known as power factor

It is said to be leading if current leads voltage, lagging if current lags voltage . Thus , a power factor of 0.5 lagging means current lags voltage by 60^O (as $\cos^{-1}0.5 = 60^O$). the product of V_{rms} and i_{rms} gives the apparent power. While the true power is obtained by multiplying the apparent power by the power factor cosφ.

(i) Resistive circuit: For $\varphi=0^O$, the current and voltage are in phase. The power is thus, maximum .

(ii) purely inductive or capacitive circuit: For $\varphi=90^O$, the power is zero. The current is then stated as wattless. Such a case will arise when resistance in the circuit is zero. The circuit is purely inductive or capacitive

(iii) *LCR series circuit*: In an *LCR* series circuit, power dissipated is given by $P = I^2Z\cos\phi$ where

$$\varphi = tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

So, ϕ may be non-zero in a *RL* or *RC* or *RCL* circuit. Even in such cases, power is dissipated only in the resistor.

(iv) *Power dissipated at resonance in LCR circuit*: At resonance $X_C - X_L = 0$, and $\phi = 0$. Therefore, $\cos\phi = 1$ and $P = I^2Z = I^2R$. That is, maximum power is dissipated in a circuit (through *R*) at resonance

Solved Numerical

Q) In an L-C-R A.C. series circuit L = 5H, ω = 100 rad s^{-1}, R = 100Ω and power factor is 0.5. Calculate the value of capacitance of the capacitor

Solution:

Power factor

$$cos\delta = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$

Squaring on both side

$$cos^2\delta = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}$$

Cosδ = 0.5

$$\frac{1}{4} = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}$$

$$R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2 = 4R^2$$

$$\left(\omega L - \frac{1}{\omega L}\right)^2 = 3R^2$$

$$\omega L - \frac{1}{\omega C} = \sqrt{3}R$$

$$\omega L - \sqrt{3}R = \frac{1}{\omega C}$$

$$C = \frac{1}{\omega}\left(\frac{1}{\omega L - \sqrt{3}R}\right)$$

$$C = \frac{1}{100}\left(\frac{1}{100 \times 5 - \sqrt{3} \times 100}\right)$$

$$C = \frac{10^{-2}}{500 - 173.2} = \frac{10^{-2}}{326.8} = 30.6 \times 10^{-6} F$$

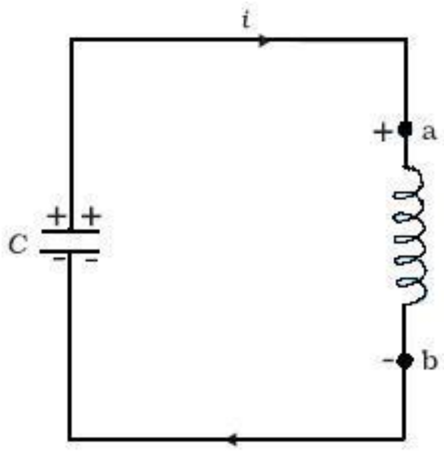
C=30.6μF

LC OSCILLATIONS

We know that a capacitor and an inductor can store electrical and magnetic energy, respectively.

When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electrical oscillations similar to oscillations in mechanical systems.

Let a capacitor be charged q_m (at t = 0) and connected to an inductor as shown in Fig..



The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let q and i be the charge and current in the circuit at time t.

Since di/dt is positive, the induced emf in L will have polarity as shown, i.e., $v_b < v_a$.

According to Kirchhoff's loop rule,

$$\frac{q}{C} - L\frac{di}{dt} = 0$$

i = – (dq/dt) in the present case (as q decreases, i increases).

Therefore, above equation becomes:

$$\frac{d^2q}{dt^2} - \frac{1}{LC}q = 0$$

Comparing above equation with standard equation for oscillation

$$\frac{d^2x}{dt^2} - \omega_0^2 x = 0$$

The charge, therefore, oscillates with a natural frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and varies sinusoidally with time as q = $q_m \cos(\omega_0 t + \varphi)$
where *qm* is the maximum value of *q* and ϕ is a phase constant.
Since *q* = *qm* at *t* = 0, we have cos ϕ =1 or ϕ = 0. Therefore, in the present case,
q = $q_m \cos(\omega_0 t)$ current I = $i_m \sin(\omega_0 t)$ here $i_m = q_m \omega_0$
Initially capacitor is fully charged , it stores energy in the form of electric field

$$U_E = \frac{1}{2} CV^2$$

At *t* = 0, the switch is closed and the capacitor starts to discharge As the current increases, it sets up a magnetic field in the inductor and thereby, some energy gets stored in the inductor in the form of magnetic energy:

$$U_B = \frac{1}{2} Li^2.$$

As the current reaches its maximum value i_m, (at *t* = *T*/4) all the energy is stored in the magnetic field:

$$U_B = \frac{1}{2} Li^2.$$

The capacitor now has no charge and hence no energy. The current now starts charging the capacitor, This process continues till the capacitor is fully charged (at *t* = *T*/2But it is charged with a polarity opposite to its initial state The whole process just described will now repeat itself till the system reverts to its original state. Thus, the energy in the system oscillates between the capacitor and the inductor.
Note that the above discussion of LC oscillations is not realistic for two reasons:
(i) Every inductor has some resistance. The effect of this resistance is to introduce a damping effect on the charge and current in the circuit and the oscillations finally die away.
(ii) Even if the resistance were zero, the total energy of the system would not remain constant. It is radiated away from the system in the form of electromagnetic waves (discussed in the next chapter). In fact, radio and TV transmitters depend on this radiation.

Solved Numerical

Q) A capacitor of capacitance 25μF is charged to 300V. It is then connected across a 10mH inductor. The resistance of the circuit is negligible
(a) Fins the frequaency of oscillation of the circuit
(b) Find the potential difference across capacior and magnitude of circuit cutrrent 1.2ms after the inductor and capacitor are connected
(c) Find the magnetic energy and electric energy at t=0 and t = 1.2 ms.

Solutions:

(a) The frequency of oscillation of the circuit is

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Substituting the given values we have

$$f = \frac{1}{2\pi\sqrt{(10 \times 10^{-3})(25 \times 10^{-6})}} = \frac{10^3}{\pi} Hz$$

(b) Charge across the capacitor at time t will be

$q = q_O \cos\omega_o t$ and $I = -q\omega_o \sin\omega_o t$

Here $q_o = CV_O = (25\times10^{-6})(300) = 7.5\times10^{-3}C$

Now, charge in the capacitor after $t = 1.25\times10^{-3}$ s is

$q = (7.5\times10^{-3})\cos(2\pi\times318.3)(1.2\times10^{-3})C = 5.53\times10^{-3}C$

∴ P.D across capacitor ,

$$V = \frac{|q|}{C} = \frac{5.53 \times 10^{-3}}{25 \times 10^{-6}} = 221.2\ volt$$

The magnitude of current in the circuit at $t = 1.2 \times10^{-3}$ s is

$|i| = q\omega_o \sin\omega_o t$

$|i| = (7.5\times10^{-3})(2\pi)(318.3)\sin(2\pi\times318.3)(1.2\times10^{-3})A = 10.13A$

(c) At t = 0, Current in the circuit is zero. Hence $U_L = 0$

Charge on the capacitor is maximum

Hence

$$U_C = \frac{1}{2}\frac{q_0^2}{C}$$

$$U_C = \frac{1}{2}\frac{(7.5 \times 10^{-3})^2}{25 \times 10^{-6}} = 1.125\ J$$

At t = 1.25ms, $q = 5.53\times10^{-3}C$

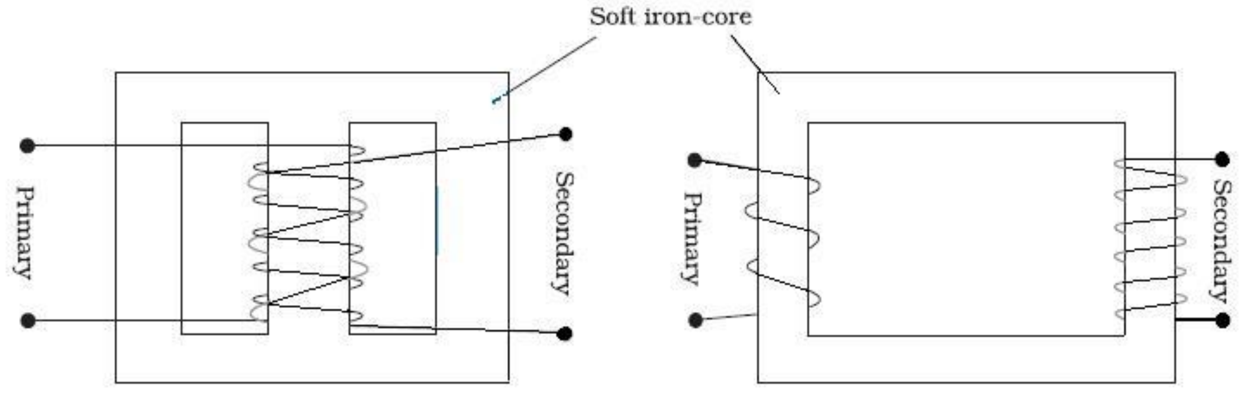
$$U_C = \frac{1}{2}\frac{q_0^2}{C}$$

$$U_C = \frac{1}{2}\frac{(5.53 \times 10^{-3})^2}{25 \times 10^{-6}} = 0.612\ J$$

TRANSFORMERS

For many purposes, it is necessary to change (or transform) an alternating voltage from one to another of greater or smaller value. This is done with a device called *transformer* using the principle of mutual induction. A transformer consists of two sets of coils, insulated from each other. They are wound on a soft-iron core, either one on top of the other as in Fig.a or on separate limbs of the core as in Fig. (b).

One of the coils called the *primary coil* has *Np* turns. The other coil is called the *secondary coil*; it has *Ns* turns. Often the primary coil is the input coil and the secondary coil is the output coil of the transformer



(a) (b)

When an alternating voltage is applied to the primary, the resulting current produces an alternating magnetic flux which links the secondary and induces an emf in it. The value of this emf depends on the number of turns in the secondary. We consider an ideal transformer in which the primary has negligible resistance and all the flux in the core links both primary and secondary windings. Let ϕ be the flux in each turn in the core at time *t* due to current in the primary when a voltage *vp* is applied to it.
Then the induced emf or voltage E*s*, in the secondary with *Ns* turns is

$$E_S = -N_S \frac{d\varphi}{dt}$$

The alternating flux φ also induces an emf, called back emf in the primary. This is

$$E_P = -N_P \frac{d\varphi}{dt}$$

But E*p* = *Vp.* If this were not so, the primary current would be infinite since the primary has zero resistance(as assumed). If the secondary is an open circuit or the current taken from it is small, then to a good approximation
E*s* = *Vs*
where *Vs* is the voltage across the secondary. Therefore, above equations can be written as

$$V_S = -N_S \frac{d\varphi}{dt}$$

$$V_P = -N_P \frac{d\varphi}{dt}$$

From above equations

$$\frac{V_S}{V_P} = \frac{N_S}{N_P}$$

Note that the above relation has been obtained using three assumptions:
(i) the primary resistance and current are small;
(ii) the same flux links both the primary and the secondary as very little flux escapes from the core, and
(iii) the secondary current is small.

If the transformer is assumed to be 100% efficient (no energy losses), the power input is equal to the power output, and since p = i V,

$$i_pV_p = i_sV_s$$

The equations obtained above apply to ideal transformers (without any energy losses). But in actual transformers, small energy losses do occur due to the following reasons:
(i) *Flux Leakage*: There is always some flux leakage; that is, not all of the flux due to primary passes through the secondary due to poor design of the core or the air gaps in the core. It can be reduced by winding the primary and secondary coils one over the other.
(ii) *Resistance of the windings*: The wire used for the windings has some resistance and so, energy is lost due to heat produced in the wire (*I* 2*R*). In high current, low voltage windings, these are minimized by using thick wire.
(iii) *Eddy currents*: The alternating magnetic flux induces eddy currents in the iron core and causes heating. The effect is reduced by having a laminated core.
(iv)Hysteresis: The magnetization of the core is repeatedly reversed by the alternating magnetic field. The resulting expenditure of energy in the core appears as heat and is kept to a minimum by using a magnetic material which has a low hysteresis loss.

The large scale transmission and distribution of electrical energy over long distances is done with the use of transformers. The voltage output of the generator is stepped-up (so that current is reduced and consequently, the I^2R loss is cut down). It is then transmitted over long distances to an area sub-station near the consumers. There the voltage is stepped down. It is further stepped down at distributing sub-stations and utility poles before a power supply of 240 V reaches our homes.

Solved Numerical

Q) In an ideal step-up transformer input voltage is 110V and current flowing in the secondary is 10A. If transformation ratio is 10, calculate output voltage, current in primary, input and out put power

Solution:
Transformer ratio

$$r = \frac{N_S}{N_P} = 10$$

(i)

$$\frac{E_S}{E_P} = \frac{N_S}{N_P}$$

$$E_S = E_P \frac{N_S}{N_P} = 110 \times 10 = 1100\ V$$

$$Ouput\ voltage\ E_S = 1100\ V$$

(ii)

$$E_P I_P = E_S I_S$$

$$I_P = \frac{E_S}{E_P} I_S = 10 \times 10 = 100\ A$$

(iii)
Input power = Output power for ideal transformer

$$E_S I_S = E_P I_p = (1100)(10) = 11000\ W$$